Homework 8
Due: Wednesday, April 11

All homeworks are due at 12:55 PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your Banner ID (but not your name or your CS login) on each page of your homework, label all work with the problem number, and staple the entire handin before submitting.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 1

Prove that for all \( n \in \mathbb{Z}^+ \),

\[
\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2
\]

Problem 2

In this problem, you will sketch another proof of Fermat’s Little Theorem using the Binomial Theorem and modular arithmetic.

As a reminder, Fermat’s Little Theorem states that for any integer \( a \) relative to a prime number \( p \),

\[
a^p \equiv a \pmod{p}
\]

and the Binomial Theorem states that, for any integers \( a, b, \) and \( n \):

\[
(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i
\]

a. Prove that, for any prime number \( p \) and any integer \( 0 < i < p \),

\( p \) divides \( \binom{p}{i} \).

b. Prove that, for integers \( a \) and \( b \) and a prime number \( p \), \( (a+b)^p \equiv a^p + b^p \pmod{p} \).

c. Use the previous part to prove Fermat’s Little Theorem via induction.

**Hint:** Use the fact that the congruence in part b holds when \( b \) is equal to 1.
Problem 3

An $n$-bit boolean function $f$ maps a 0/1 string of length $n$ to either 0 or 1.

a. Count the number of $n$-bit boolean functions. Write your expression in terms of $n$.

b. How many $n$-bit boolean functions are bijective?

An $n$-bit boolean function *depends* on the position $i$ in the string if there exists two strings $A$ and $B$ such that $A$ and $B$ differ only in position $i$ and $f(A) \neq f(B)$.

c. Count the number of $n$-bit boolean functions that do not depend on bit $i$.

d. For $n > 1$, count the number of $n$-bit boolean functions that map a string to 1 if (not iff) its first bit, second bit, or both are 1.

Let $A = (a_1, a_2, \ldots, a_n) \in \{0, 1\}^n$ be a 0/1 string of length $n$ where $a_i$ represents the bit at position $i$. A boolean function $f$ is *symmetric* if for all $A$ in $\{0, 1\}^n$, $f$ maps $A$ and all of the strings corresponding to each of the possible permutations of $a_1, a_2, \ldots, a_n$ to the same element in the codomain.

e. Count the number of $n$-bit symmetric boolean functions.

Problem 4

Beret Guy is brushing up on his math skills. He is given the following equation:

\[ x_1 + x_2 + x_3 + x_4 = 100 \]

where each $x_i$ must be non-negative. After seeing Beret Guy’s questionable arithmetic, Megan decides to help him by counting the number of solutions to his equation.

a. Count the number of solutions to this equation.

b. Now suppose we require a solution with $x_1$ and $x_4$ strictly positive. Count the number of solutions under this new constraint.

c. More generally, suppose we require a solution where $a_i$ is a fixed constant non-negative integer and $x_i \geq a_i$, satisfying

\[ \sum_{i=1}^{4} a_i \leq 100 \]

Again, count the number of solutions under this new constraint.
See xkcd comic number 759 for Beret Guy’s concerning attempt at calculating $3 \cdot 9$.

**Problem 5**

Denote $[n] = \{1, 2, \ldots, n\}$.

a. How many subsets of $[n]$ don’t contain two consecutive integers?

b. Suppose we have a set $S$ of subsets of $[n]$ such that each pair of subsets has at least one element in common. What is the maximum possible size of $S$?