Problem 1

Consider the following inductive proof.

Claim: For any group of \( n \geq 1 \) people in the world, everyone will have the same eye color.

Proof. Proof is by induction.

Base case: When \( n = 1 \), clearly everyone in the room has the same eye color as there is only one person in the room.

Inductive Hypothesis: Assume that in a group of \( k \) people, everyone in the group has the same eye color.

Inductive Step: We will now show that in a group of \( k + 1 \) people, everyone has the same eye color.

Consider an arbitrary ordering of the \( k + 1 \) people. That is, assign each person in the room a unique number from 1 to \( k + 1 \). We know from our inductive hypothesis that Person 1 through Person \( k \) have the same eye color. Also from our inductive hypothesis, Person 2 through Person \( k + 1 \) have the same eye color. Since there is some overlap between the first \( k \) people and the last \( k \) people, everyone in the room must have the same eye color.

Clearly something is wrong with this proof. Explain the flaw in this inductive proof. You should be able to explain the flaw in just a few lines.
Problem 2

Let $p, q$ be consecutive odd primes, with $p < q$. Prove that $p + q$ is a product of 3 integers, each greater than 1.

Problem 3

For any $m \in \mathbb{Z}^+$, define the relation $R_m$ on $\mathbb{Z}$ by

$$R_m = \{(x, y) \mid x \text{ and } y \text{ have the same remainder when divided by } m\}.$$

a. Prove that $R_m$ is an equivalence relation.

b. Describe the equivalence classes of $R_m$. How many are there (in terms of $m$)?

Problem 4

Recall that a partial order is a relation that is reflexive, antisymmetric, and transitive. Prove that the divisibility relation $\mathcal{R} = \{(a, b) \mid a \text{ divides } b\}$ on the positive integers is a partial order.

Problem 5

White Hat and Black Hat have become bored during yet another franchise meeting, and White Hat proposes that they play a game. He gathers stray paperclips lying around his desk and separates them into two piles of equal size. At each turn, White Hat and Black Hat each remove some non-zero number of paperclips from one of the piles. The player who removes the last paperclips wins.

Since the game is White Hat’s idea, he decides to go first. Prove that Black Hat can always win.