Homework 3
Due: February 23, 2017 at 12:55pm

All homeworks are due at 12:55 PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your Banner ID (but not your name or your CS login) on each page of your homework, label all work with the problem number, and staple the entire handin before submitting.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 1

Let $A$ be a set with $n$ elements. Let $T$ be the set of all ordered pairs $(X,Y)$ where $X$ and $Y$ are subsets of $A$. Let $S$ be the set of 0/1/2/3 strings of length $n$. That is, elements of $S$ are strings of length $n$ where each character is 0, 1, 2, or 3. Give (and prove) a bijection between $T$ and $S$.

Conclude that $T$ and $S$ must be the same size.

Problem 2

a. Prove the following statement by induction.

$$\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

b. Consider a fun two-player game in which there are initially two nonempty piles of spoons. At each turn, players choose a pile and take a positive number of spoons from that pile. Players alternate turns, and whoever removes the last spoon wins the game. Prove that if the two piles contain the same number of spoons initially, the second player can always guarantee a win.

Problem 3

Prove by induction that any positive integer $n$ can be written as:

$$a_k3^k + a_{k-1}3^{k-1} + \ldots + a_13^1 + a_0$$

where each $a_i \in \{-1, 0, 1\}$.
For example:

- $8 = 1 \cdot 3^2 + 0 \cdot 3^1 + (-1) \cdot 3^0$
- $19 = 1 \cdot 3^3 + (-1) \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0$

**Problem 4**

Suppose $R$ is a relation on a set $A$ and $q$ is a property of relations (such as reflexivity, symmetry, or transitivity). For property $q$, define the $q$-closure of $R$ as the smallest relation on $A$ that has property $q$ and is a superset of $R$. A set $S$ is a superset of $T$ if and only if $T$ is a subset of $S$.

For example, let $A = \{1, 2, 3\}$ and $R = \{(2, 1)\}$. The reflexive closure of $R$ is the relation $\{(2, 1), (1, 1), (2, 2), (3, 3)\}$ while the symmetric closure of $R$ is the relation $\{(2, 1), (1, 2)\}$.

a. Given a relation $R$, is the transitive closure of the symmetric closure of its reflexive closure an equivalence relation? (i.e. If one took the reflexive closure of $R$, and then the symmetric closure of the result, and then took the transitive closure of that, is it an equivalence relation?)

b. Given a relation $R$, is the reflexive closure of the symmetric closure of its transitive closure an equivalence relation?

**Problem 5**

a. Prove by contradiction that for any integer $n$, $n^2 - 2$ is not divisible by 4.

b. Prove that for any integer $n$, $n^3$ is odd if and only if $n$ is odd.

c. Prove that for any integer $n$, $n^3 - n$ is divisible by 3.