Homework 2
Due: February 15, 2017 at 12:55pm

All homeworks are due at 12:55 PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your Banner ID (but not your name or your CS login) on each page of your homework, label all work with the problem number, and staple the entire handin before submitting.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 1

Consider the following relation on the set of integers:

\[ \forall a, b \in \mathbb{Z}, \ (a, b) \in R \text{ if and only if the remainder when } a \text{ is divided by 3 is the same as the remainder when } b \text{ is divided by 3.} \]

a. Prove that \( R \) is an equivalence relation.

b. How many distinct equivalence classes are in this equivalence relation? What are they?

Solution

a. **Reflexive** \( \forall a \in \mathbb{Z}, \ (a, a) \in R \). Let the remainder when \( a \) is divided by 3 be \( \ell \). Since \( \ell = \ell \), we can say that \( (a, a) \in R \) by definition.

**Symmetric** Let \( \ell \) and \( m \) be the remainders for \( a \) and \( b \) when divided by 3, respectively. Given \( (a, b) \in R \), we know that \( \ell = m \). Thus \( m = \ell \), so \( (b, a) \in R \) by definition.

**Transitive** Let \( \ell \), \( m \), and \( n \) be the remainders for \( a \), \( b \), and \( c \) when divided by 3, respectively.

Given \( (a, b) \in R \) and \( (b, c) \in R \), we know that \( \ell = m \) and \( m = n \). Thus \( \ell = n \).

Since \( \ell = n \), we know that \( (a, c) \in R \) by definition of our equivalence relation.

**Conclusion** Since the relation \( R \) is reflexive, symmetric and transitive, \( R \) is
an equivalence relation by definition.

b. There are three equivalences classes for this relation.

- \([1]_R\) or equivalent
- \([2]_R\) or equivalent
- \([3]_R\) or equivalent

Problem 2

A relation \(R\) is a *partial order* over a set \(S\) if and only if it satisfies the following properties:

**Reflexivity**: \((x, x) \in R\) for all \(x \in S\).

**Antisymmetry**: If \((x, y) \in R\) and \((y, x) \in R\), then \(x = y\).

**Transitivity**: If \((x, y) \in R\) and \((y, z) \in R\), then \((x, z) \in R\).

Ben and Jerry love experimenting with new ice cream flavors. They have noticed that the more ingredients they add, the fancier the flavor seems to get!

A flavor, \(f\), is a set of ingredients. We define the relation \(F\) over all possible flavors, where

\[
F = \{(f_1, f_2) \mid \text{each ingredient in } f_1 \text{ is also in } f_2\}.
\]

a. Prove that \(F\) is a partial order over the set of all ice cream flavors. (Assume that a flavor with no ingredients is still a flavor.)

b. Hasse diagrams are a convenient way to represent finite partial orderings. To draw a Hasse diagram for partial ordering \(R\), we write down each element in the set. Draw an arrow from \(x\) to \(z\) if \(xRz\) and if there exists no \(y\) such that \(xRy\) and \(yRz\).

Let \(I = \{\text{milk, sugar, cookies}\}\) be the set of ingredients that Ben and Jerry have. Draw a Hasse diagram for \(F\).

Solution

a. We will prove that \(F\) is a partial order over the set of all ice cream flavors (\(\mathcal{P}(I)\)) by proving that \(F\) is reflexive, antisymmetric, and transitive.

**Reflexive** \(fFf\) for all \(f \in \mathcal{P}(I)\) because \(f \subseteq f\) always holds.
Antisymmetric Suppose $f_1 \subseteq f_2$ and $f_2 \subseteq f_1$. Thus $f_1 = f_2$. Therefore, $\mathcal{F}$ is antisymmetric.

Transitive Suppose $f_1 \subseteq f_2$ and $f_2 \subseteq f_3$. That is, $f_1 \subseteq f_2$ and $f_2 \subseteq f_3$. $f_1 \subseteq f_3$ by the transitivity of $\subseteq$. Therefore, $f_1 \subseteq f_3$.

We conclude that $\mathcal{F}$ is a partial order.

b. Below is the Hasse diagram.

![Hasse diagram]

Problem 3

Prove that any relation that is both an equivalence relation and a partial ordering is the identity relation. That is, if $X$ is a set and $R$ is a relation on $X$ that is both a partial ordering and an equivalence relation, then $aRb$ if and only if $a = b$.

Solution

Claim: For some relation $R$ that is both an equivalence relation and partial order over the set $S$, $xRy$ if and only if $x = y$.

Proof. We need to show both directions.
a. **Claim:** If \( xRy \) then \( x = y \).

   **Proof.** Let \( x, y \in S \) such that \( xRy \).

   By the symmetry of \( R \), \( yRx \). Since we have both \( xRy \) and \( yRx \), by antisymmetry of \( R \), \( x = y \).

b. **Claim:** If \( x = y \) then \( xRy \).

   **Proof.** Let \( x, y \in S \) such that \( x = y \). \( xRx \) because \( R \) is reflexive. Since \( x = y \), we have \( xRy \).

As we have shown that \( xRy \implies x = y \) and \( x = y \implies xRy \), we can conclude that \( xRy \) if and only if \( x = y \).

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**Problem 4**

Peter likes to buy a milkshake at the end of every weekday to reward himself for showing up to class.

Let \( F = \{\text{vanilla, chocolate, coffee, strawberry}\} \) be the set of milkshake flavors from which Peter selects. Let \( D = \{\text{Monday, Tuesday, Wednesday, Thursday, Friday}\} \) be the set of days on which Peter has class.

For each of the following relations on \( D \) and \( F \), state whether the relation is a function from \( D \) to \( F \). If the relation is not a function, explain why not. If the relation is a function, answer the following:

- What is the range of the function?
- What is the inverse image of “vanilla” under the function?
- Is the function injective?
- Is the function surjective?
- Is the function bijective?

a. \( \{(\text{Monday, vanilla}), (\text{Tuesday, coffee}), (\text{Thursday, vanilla})\} \)

b. \( \{(\text{Monday, chocolate}), (\text{Tuesday, chocolate}), (\text{Wednesday, chocolate}), (\text{Thursday, chocolate}), (\text{Friday, vanilla})\} \)
c. \{(Monday, strawberry), (Tuesday, coffee), (Wednesday, chocolate), 
(Thursday, coffee), (Friday, vanilla)\}

d. \{(Monday, vanilla), (Wednesday, chocolate), (Thursday, coffee), (Thursday, strawberry)\}

**Solution**

a. This set does not determine a function, because not every input value in \(D\) has an output in \(F\). For instance, “Wednesday” is not paired with an output value.

b. This set determines a function.

- The range is \{vanilla, chocolate\}.
- The inverse image of “vanilla” is \{Friday\}.
- This function is not injective, because e.g. “Monday” and “Tuesday” both map to “chocolate”.
- This function is not surjective, because e.g. “coffee” is not mapped to.
- This function is not bijective because it is not injective.

c. This set determines a function.

- The range is \{vanilla, chocolate, coffee, strawberry\}.
- The inverse image of “vanilla” is \{Friday\}.
- This function is not injective, because “Tuesday” and “Thursday” both map to “coffee”.
- This function is surjective because each member of \(F\) has at least one matching value from \(D\).
- This function is not bijective since it is not injective.

d. This set does not determine a function, because there are two outputs for one input: “Thursday” maps to two distinct values.

**Problem 5**

Jay Curley is the Senior Global Marketing Manager at Ben and Jerry’s. He is responsible for deciding which stores carry which flavors.

Consider the nonempty set of flavors \(F = \{f_1, f_2, \ldots, f_n\}\) that a store can carry.

\(^1\)https://www.linkedin.com/in/jaycurley
Jay can tell a store to carry any subset of these flavors (even the empty set). He decides to assign each of these subsets to exactly one store.

Prove that the number of stores that sell an even number of flavors is equal to the number of stores that sell an odd number of flavors. Prove this by constructing a bijection between the set of stores that sell an even number of flavors and the set of stores that sell an odd number of flavors.

Be sure to prove that your function is in fact bijective.

**Solution**

We will call the set of stores that sell an even number of flavors $E$, and the set of stores that sell an odd number of flavors $O$.

In order to prove that $|E| = |O|$, we will construct a bijection $f$ between $O$ and $E$ as follows:

$$f(X) = \begin{cases} X \setminus \{a\} : a \in X \\ X \cup \{a\} : a \notin X \end{cases}$$

where $X \in O$ and $a$ is some arbitrary but fixed element of $F$. Note that each set of cardinality $2k + 1$ is mapped to a set of cardinality $2k$ (by removing an element) or $2k + 2$ (by adding an element.)

First, we prove that $f$ is injective. For the sake of contradiction, suppose that there exist $X, Y \in O$ such that $X \neq Y$ but $f(X) = f(Y)$. Since $X \neq Y$, $X$ and $Y$ must differ by at least one element. We examine the three different cases:

- $a$ is an element in both sets. This is a contradiction because $X \neq Y$ implies $X \setminus \{a\} \neq Y \setminus \{a\}$.
- $a$ is an element in neither set. This is a contradiction because $X \neq Y$ implies $X \cup \{a\} \neq Y \cup \{a\}$.
- $a$ is an element in one set but not the other. Without loss of generality, suppose $a \in X$ and $a \notin Y$. There is a contradiction: it cannot be the case that $f(X) = f(Y)$ because $a \notin f(X)$ and $a \in f(Y)$.

Therefore, $X \neq Y$ implies $f(X) \neq f(Y)$, meaning that $f$ is injective, as needed.

Next, we prove that $f$ is surjective. Consider an arbitrary element $Z \in E$. We can construct an $X$ such that $f(X) = Z$ as follows:
• if $a \in Z$, let $X = Z \setminus \{a\}$
• if $a \notin Z$, let $X = Z \cup \{a\}$

It must be the case that $X \in \mathcal{O}$, since $X$ must have odd cardinality (it was constructed by either adding one element or removing one element from a set of even cardinality). Thus, since all $Z \in \mathcal{E}$ are mapped to by some $X \in \mathcal{O}$, $f$ is surjective.

Since we can create a bijection between $\mathcal{E}$ and $\mathcal{O}$, we can conclude that $|\mathcal{E}| = |\mathcal{O}|$. 