Homework 1

Due: February 8, 2017 at 12:55pm

All homeworks are due at 12:55 PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your Banner ID (but not your name or your CS login) on each page of your homework, label all work with the problem number, and staple the entire handin before submitting.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 1

The set difference of two sets $A$ and $B$ is defined to be all elements in set $B$ that are not in $A$, written as follows:

$$ B \setminus A = \{ x \in B \mid x \not\in A \}.$$

For this problem, let

- $A = \{\emptyset, 0, \{\emptyset\}, \{0, \emptyset\}\}$
- $B = \{\emptyset, \{\emptyset\}, \{0, \emptyset, 1\}\}$
- $C = \{ x \mid \exists y \in \mathbb{Z} \text{ s.t. } y^2 = x, \ x < 10 \}$
- $D = \{a, b, c, d, e\}$
- $E = \{c, d, e\}$
- $F = \{d, e, f, g\}$

a. Find the following sets:

i. $A \cup B$

ii. $A \cap B$

iii. $A \setminus B$

iv. $P(B) \ (= 2^B)$, the power set of $B$

v. $C \setminus (A \cup B)$

vi. $\{x \mid x \in B, |x| \notin C\}$

vii. $A \times E$
viii. \((D \times E) \setminus (E \times F)\)
ix. \((D \setminus E) \cap (D \cap E)\)

b. Find the cardinalities of the following sets:

i. \(C\)
ii. \(A \times B\)
iii. \(\mathcal{P}(A \cup B)\)
iv. \(\mathcal{P}(\mathcal{P}(\emptyset))\)
v. \(F \setminus (D \cap E)\)

Solution

a. i. \(\{\emptyset, 0, \{\emptyset\}, \{0, \emptyset\}, \{0, \emptyset, 1\}\}\)
   ii. \(\{\emptyset, \{\emptyset\}\}\)
   iii. \(\{0, \{0, \emptyset\}\}\)
   iv. \(\left\{ \emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\emptyset, 1\}\}, \{\emptyset, \{\emptyset, 1\}\}, \{\{\emptyset\}, \{0, \emptyset, 1\}\}, \{\emptyset, \{\emptyset\}, \{0, \emptyset, 1\}\} \right\}\)
   v. \(\{1, 4, 9\}\)
   vi. \(\{\{0, \emptyset, 1\}\}\)

vii. \(\{(\emptyset, c), (\emptyset, d), (\emptyset, e), (0, c), (0, d), (0, e), (\{\emptyset\}, c), (\{\emptyset\}, d), (\{\emptyset\}, e), (\{0, \emptyset\}, c), (\{0, \emptyset\}, d), (\{0, \emptyset\}, e)\}\)

viii. \(\{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, c), (d, c), (e, c)\}\)
ix. \(\emptyset\)

b. i. 4
ii. 12
iii. 32
Problem 2

Due to an abundance of new flavors, Ben needs to choose some of the current flavors to keep in stores and must send the rest to the Flavor Graveyard. He wants to keep an ice cream flavor if (and only if): (i) either he finds them delicious or they are popular, and (ii) they are not celebrity sponsored.

Consider the following sets:

Ice Creams: \( I = \{ \text{Americone Dream, Cherry Garcia, Chocolate Therapy, Chunky Monkey, Half Baked, Phish Food, The Tonight Dough} \} \)

Delicious: \( D = \{ \text{Chocolate Therapy, Chunky Monkey, Half Baked} \} \)

Popular: \( P = \{ \text{Cherry Garcia, Chocolate Therapy, The Tonight Dough} \} \)

Sponsored: \( S = \{ \text{Americone Dream, The Tonight Dough} \} \)

For parts a through e, write the phrase using \( \cup, \cap, \text{ and } \setminus \) in proper set notation (in terms of the above four sets) and list the elements in the set. For example, the answer for “delicious ice creams” would be

\[ I \cap D = \{ \text{Chocolate Therapy, Chunky Monkey, Half Baked} \} \]

Be sure to specify the order of set operations with parentheses where needed.

a. Ice creams that are not popular.
b. Popular flavors that are not delicious.
c. Ice creams that are either delicious or sponsored.
d. Popular flavors that are delicious and ice cream.
e. The list of flavors of ice cream that Ben wants to keep.

Solution

a. \( I \setminus P = \{ \text{Americone Dream, Chunky Monkey, Half Baked, Phish Food} \} \)
b. \( P \setminus D = \{ \text{Cherry Garcia, The Tonight Dough} \} \)
c. \( I \cap (D \cup S) = \{ \text{Americone Dream, Chocolate Therapy, Chunky Monkey, Half Baked, The Tonight Dough} \} \)
d. \( P \cap D \cap I = \{ \text{Chocolate Therapy} \} \)
Problem 3

Assume that all sets are subsets of a universal set $U$. Use the element method to prove the following:

a. For all sets $A$ and $B$, $A \cap B = (A \cup B) \setminus (A^c \cup B^c)$

b. For all sets $A$ and $B$, $A \cup (B \setminus A) = A \cup B$.

c. For all sets $A$, $B$, and $C$, $(A \setminus C) \cap (B \setminus C) \cap (A \setminus B) = \emptyset$.

Solution

a. Suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$. So $x \in A \cup B$. Also $x \notin A^c$ and $x \notin B^c$ by the definition of complement.

Since $x$ is in neither $A^c$ nor $B^c$, $x \notin (A^c \cup B^c)$. So $x \in (A \cup B) \setminus (A^c \cup B^c)$. Thus, $A \cap B$ is a subset of $(A \cup B) \setminus (A^c \cup B^c)$.

Suppose $y \in (A \cup B) \setminus (A^c \cup B^c)$. Then $y \in A \cup B$ and $y \notin (A^c \cup B^c)$. So $y \notin A^c$ and $y \notin B^c$. So $y \in A$ and $y \in B$ by the definition of complement. So $y \in A \cap B$. Thus, $(A \cup B) \setminus (A \cap B)$ is a subset of $A \cap B$.

Therefore, $A \cap B = (A \cup B) \setminus (A^c \cup B^c)$.

b. Let $x \in A \cup (B \setminus A)$. If $x \in A$, then $x \in A \cup B$. If $x \in B \setminus A$, then $x \in B$ but $x \notin A$, so $x \in B$, meaning that $x \in A \cup B$. This proves $A \cup (B \setminus A) \subseteq A \cup B$.

For the other direction, suppose $y \in A \cup B$. Consider two cases: either $y \in A$, or $y \notin A$. In the first case, $y \in A \cup (B \setminus A)$ as desired. In the second case, $y \in A \cup B$, and $y \notin A$, so $y \in B$. Since $y \in B$ and $y \notin A$, $y \in B \setminus A$. So $y \in A \cup (B \setminus A)$. Therefore $A \cup B \subseteq A \cup (B \setminus A)$. □

c. We need to show that for all $x \in (A \setminus C) \cap (B \setminus C) \cap (A \setminus B)$, $x \in \emptyset$, and that for all $x \in \emptyset$, $x \in (A \setminus C) \cap (B \setminus C) \cap (A \setminus B)$. There are no elements of the empty set, so the second statement is vacuously true.

For the first statement, consider $x \in (A \setminus C) \cap (B \setminus C) \cap (A \setminus B)$.

This means that $x \in (A \setminus C)$, $x \in (B \setminus C)$, and $x \in (A \setminus B)$.

Simplifying this we get that $x \in A$, $x \in B$, $x \in C^c$, and $x \in B^c$. This tells us that $x \in (B \cap B^c)$ which can be equivalently written as $x \in \emptyset$. 
This means that \((A \setminus C) \cap (B \setminus C) \cap (A \setminus B) \subseteq \emptyset\).

Thus, we have proven that \((A \setminus C) \cap (B \setminus C) \cap (A \setminus B) = \emptyset\).

**Problem 4**

a. Prove that a relation that is reflexive and symmetric need not also be transitive.

b. Prove that a relation that is reflexive and transitive need not also be symmetric.

c. Prove that a relation that is symmetric and transitive need not also be reflexive.

**Solution**

Proofs are by counterexample. Let \(S\) be a finite set, and let \(R\) be a relation on \(S\):

a. \(S = \{a, b, c, d\}\) and \(R = \{(a, a), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c), (d, d)\}\).

We will show that \(R\) is reflexive and symmetric, but not transitive.

\(R\) is reflexive because for all elements in \(S\), \((a, a), (b, b), (c, c), (d, d) \in R\).

\(R\) is symmetric because for all elements in \(R\), the reversed tuple is also in \(R\): that is, \((a, c), (c, a) \in R\), \((b, c), (c, b) \in R\), and \((a, a), (b, b), (c, c), (d, d) \in R\).

\(R\) is not transitive because \((a, c), (c, b) \in R\) but \((a, b) \notin R\).

b. \(S = \mathbb{Z}\) and \(R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \leq y\}\).

We will show that \(R\) is reflexive and transitive, but not symmetric.

\(R\) is reflexive, because for all \(a \in S\), \(a \leq a\) and \(a \leq a\) hold. So \((a, a) \in R\).

\(R\) is transitive, because if \(a, b, c \in S\) and \((a, b) \in R\) and \((b, c) \in R\), then \(a \leq b\) and \(b \leq c\). So \(a \leq c\). Thus, \((a, c) \in R\).

\(R\) is not symmetric, because \(1 \leq 2\), but \(2 \not\leq 1\).

(In fact, \(R\) is antisymmetric, thereby making it a partial order. We leave as an exercise to the reader.)

c. \(S = \{a, b, c\}\) and \(R = \{\}\).

We will show that \(R\) is symmetric and transitive, but not reflexive.

\(R\) is vacuously symmetric. That is, because \(R\) is empty, it is never the case that \((x, y) \in R\) for any \(x, y \in S\). Therefore, symmetry is never contradicted.

\(R\) is transitive by an analogous argument.
$R$ is not reflexive, because it is not the case that for every element $s$ in $S$, $(s, s) \in R$. In fact, this does not hold for any $s \in S$.

Note: $R$ does not have to be nonempty to make a counterexample. For example, another counterexample is $A = \{a, b, c\}$ and $R = \{(a, a), (a, b), (b, a), (b, b)\}$.

**Problem 5**

Prove that the sum of any four consecutive integers is even.

**Solution**

Let $a \in \mathbb{Z}$. The next three consecutive integers are $a + 1$, $a + 2$, and $a + 3$ respectively.

\[
\begin{align*}
    a + (a + 1) + (a + 2) + (a + 3) &= 4a + 1 + 2 + 3 \\
    &= 4a + 6 \\
    &= 2(2a + 3)
\end{align*}
\]

This is of the form $2$ times some integer. Therefore, the sum of any four consecutive integers is even.

**Problem 6**

Prove the following statement by contradiction:

For all real numbers $x$ and $y$, if $x$ is irrational and $y$ is rational then $x - y$ is irrational.

**Solution**

Let $x$ be irrational and $y$ be rational. Assume $x - y$ is rational.

Since $y$ is rational, $\exists n, m \in \mathbb{Z}$ such that $y = \frac{n}{m}$.

Since $x - y$ is rational, $\exists p, q \in \mathbb{Z}$ such that $x - y = \frac{p}{q}$.

Then $x = (x - y) + y = \frac{p}{q} + \frac{n}{m} = \frac{pm + nq}{qm}$, meaning $x$ must be rational.

However, that is a contradiction to our assumption that $x$ is irrational. Therefore,
for all real numbers $x$ and $y$, if $x$ is irrational and $y$ is rational, then $x - y$ is irrational.