Existence of a Nash equilibrium.

Consider a game with players \( \{1, 2, \ldots, I\} \), where each player \( i \) has a finite nonempty set \( S_i \) of possible pure strategies, and a utility function \( u_i : S \rightarrow \mathbb{R} \), from the set of (pure) strategy profiles \( S = \prod_i S_i \) to the reals. A mixed strategy is a distribution over pure strategies, leading to the notion of mixed strategy profiles and to expected utility.

A strategy profile \( \sigma = (\sigma_1, \ldots, \sigma_I) \) is a Nash equilibrium if for every player \( i \), and every mixed strategy \( \sigma'_i \), the expected utility of \( i \) for \( (\sigma'_i, \sigma_{-i}) \) is no greater than the expected utility of \( i \) for \( \sigma \). Here we use the notation \( \sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_I) \) to denote, in profile \( \sigma \), the strategies of all the players other than player \( i \).

There does not always exist a pure Nash equilibrium.

**Theorem 1 (Nash, 1951)** There exists a mixed Nash equilibrium.

Here is a short self-contained proof.

We will define a function \( \Phi \) over the space of mixed strategy profiles. We will argue that that space is compact and that \( \Phi \) is continuous, hence the sequence define by: \( \sigma(0) \) arbitrary, \( \sigma(n) = \Phi(\sigma(n-1)) \), has an accumulation point. We will argue that every fixed point of \( \Phi \) must be a Nash equilibrium, hence the proof.

The space of mixed strategy profiles is clearly compact, since it can be described as:

\[
\{(\alpha_i(s_i)) : \forall i, \sum_{s_i \in S_i} \alpha_i(s_i) = 1; \forall i, \forall s_i \in S_i, 0 \leq \alpha_i(s_i) \leq 1\}.
\]

Given a mixed strategy profile \( \alpha = (\alpha_i(s_i)) \), the expected utility of player \( i \) is (extending the function \( u_i \) to mixed strategies)

\[
u_i(\alpha) = \sum_j \sum_{s_j \in S_j} \alpha_j(s_j) u^i((s_j)_{j \neq i}).
\]

The expected utility of player \( i \) if he were to play a particular pure strategy \( s \in S_i \) instead of \( (\alpha_i(s_i))_{s_i} \) would be

\[
u_i(s, \sigma_{-i}) = \sum_{j \neq i} \sum_{s_j \in S_j} \alpha_j(s_j) u^i(s, (s_j)_{j \neq i}).
\]

For \( s \in S_i \), let \( p_i(s, \alpha) = u_i(s, \sigma_{-i}) - u_i(\alpha) \). The function \( \Phi \) will modify the mixed strategy of player \( i \) by shifting some of the weight of the distribution to give more weight to the set of strategies \( s \in S_i \) for which \( p_i(s) > 0 \), as follows: \( \Phi(\alpha) = \alpha' \), with

\[
\alpha'_i(s_i) = \frac{\alpha_i(s_i) + \max(p_i(s_i, \alpha), 0)}{1 + \sum_{s \in S_i} \max(p_i(s, \alpha), 0)}.
\]

Clearly, \( \Phi \) is continuous. Finally, it is easy to see that

\[
\sum_{s : p_i(s, \alpha) > 0} \alpha'_i(s) = \sum_{s : p_i(s, \alpha) > 0} \frac{\alpha_i(s) + p_i(s, \alpha)}{1 + \sum_{s' : p_i(s', \alpha) > 0} p_i(s', \alpha)} \geq \sum_{s : p_i(s, \alpha) > 0} \alpha_i(s),
\]

with equality achieved only if \( p_i(s, \alpha) \leq 0 \) for every \( s \).

Every fixed point of \( \Phi \) must have \( \sum_{s : p_i(s, \alpha) > 0} \alpha'_i(s) = \sum_{s : p_i(s, \alpha) > 0} \alpha_i(s) \) for every \( i \), hence must have \( p_i(s, \alpha) \leq 0 \) for every \( i \) and every \( s \in S_i \), hence must be a Nash equilibrium. This concludes the proof of the existence of a Nash equilibrium.