The material presented this week is nearly identical to two lectures from Stanford’s second-semester Mechanism Design course, CS 364B. Rather than copying the material, we give a high-level overview of what we covered in class, together with references to sections of the CS 364B notes for deeper coverage. The CS 364B notes can be found here:

- **Lecture 2: Unit Demand Bidders and Walrasian Equilibria**
- **Lecture 3: The Crawford-Knoer Auction**

We studied the *unit demand* setting, in which bidders are only interested in acquiring one item, but have preferences for which item that is. We showed that the *winner determination problem*\(^1\) can be solved in polynomial time by representing bidders and items as a bipartite graph and solving for the maximum-weight matching. To compute VCG payments, we remove each bidder from the graph and re-run the matching to compute externalities. (Lecture 2, pp. 1–3)

Next, we presented the Crawford-Knoer (CK) auction, an ascending auction for the unit demand setting. The auction is formally described in Lecture 3, pp. 2–3. As usual, our goal is to prove that the auction is EPIC and welfare-maximizing. Recalling the strategy outlined in EPIC Mechanisms and VCG, we do this in two steps:

- Show that the CK auction terminates at the VCG outcome.
- Show that inconsistent bidding does not improve on sincere bidding.

The second step is relatively straightforward once we’ve established a couple of necessary lemmas. Showing that the auction terminates at the VCG outcome, however, is rather involved, and requires an intermediate step: we first show that the CK auction terminates at the smallest *Walrasian Equilibrium* (WE), then show that the VCG outcome is also the smallest WE. Recall that a Walrasian Equilibrium (Lecture 2, p. 4) is an allocation and price vector such that each bidder ends up with their “favorite” good at the current prices, and a good is unallocated only if its price is 0.

The following sequence of theorems—the “meat” of last class and the notes—proves the two intermediate steps and shows that the CK auction is EPIC:

1. The payments in a WE are lower-bounded by the VCG payments (Lecture 2, Theorem 3.5).
(2) The VCG outcome is a WE (Lecture 2, Theorem 3.6). This step makes use of Lemma 3.7 in Lecture 2. This is a detail we skipped over in class, so we encourage you to read the notes to fill in the gap.

(3) The CK auction terminates at a WE (Lecture 3, Lemma 3.2)—this one is pretty straightforward.

(4) The payments in the CK auction are no larger than the payments of any WE: we hand-waved over this in class. The formal proof is a bit gnarly and is given in Lecture 3, Lemma 3.4².

(5) Inconsistent bidding does not improve on sincere bidding (Lecture 3, Theorem 4.2; uses Lemma 3.5).

Observe that steps (1) and (2) together show that VCG is the smallest WE. Then (3) and (4) together show that the outcome of the CK auction is the smallest WE, so by transitivity the CK auction terminates at the VCG outcome. Step (5) is the last piece required to prove EPIC. While it is great if you understand the proofs of these theorems, at least convince yourself that their union proves what we want: that the CK auction is EPIC and welfare-maximizing.

You may have noticed that there is one theorem we proved in class that has not been mentioned, namely, the First Welfare Theorem.

Theorem. If \((q, M)\) is a Walrasian Equilibrium, then \(M\) is a welfare-maximizing allocation.

The proof, found on page 6 of Lecture 2, is relatively straightforward, yet the theorem is very powerful. It is a special case of the First Fundamental Theorem of Welfare Economics, which shows that competitive markets yield efficient resource allocation³, giving support to Adam Smith’s “invisible hand” hypothesis. For us, it helps prove the following lemma, which is necessary in the proof of (1):

Lemma. If \((q, M)\) is a Walrasian Equilibrium, and \(M^*\) is a welfare-maximizing allocation, then \((q, M^*)\) is a Walrasian Equilibrium.

The proof of this lemma is left as a homework question.

² Confusingly, the statement of the lemma is much stronger: that VCG payments upper-bound the CK payments. While this is true when you pull together all the previous steps, all the proof shows (and all that is necessary for this step) is that the CK payments are no larger than those of any WE.

³ Technically Pareto-efficient, which is weaker than the welfare-maximizing notion we use.