 Revenue Maximization in the Sponsored Search Auction

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Myerson’s formula tells us that we can maximize revenue by maximizing virtual welfare, so this optimal solution will look almost identical to the welfare solution.

- Sort bidders by virtual valuation, so that \( \varphi_1(v_1) \leq \varphi_2(v_2) \leq \cdots \leq \varphi_{n-1}(v_{n-1}) \leq \varphi_n(v_n) \).

- Assign the lowest \( n - k \) bidders an allocation of 0, as well as any bidders who had negative virtual value. This leaves \( m \) bidders, where \( m \leq n \).

- Assign bidder \( m - j + 1 \) slot \( k - j + 1 \) for \( 1 \leq j \leq k \).
  - \( m \) gets slot \( k \)
  - \( m - 1 \) gets slot \( k - 1 \)
  - \( m - 2 \) gets slot \( k - 2 \)
  - \( \ldots \)
  - \( m - k + 1 \) gets slot 1

**Remark 0.1.** Since \( m \) is only lower-bounded by 0 (since \( m = 0 \) if all bidders had negative virtual value), it is possible that \( m < k \), in which case we only assign the top \( m \) slots.

Note that this allocation scheme is also monotone by symmetric reasoning as in the welfare maximizing case. Therefore we can again apply Myerson’s Lemma. We compute the payment for the winner of the \( i \)th slot, whom we shall call bidder \( i \). To simplify notation, we let

\[
 b_j = \varphi_i^{-1}(\varphi_j(v_j))^1, 
\]

the bid required by bidder \( i \) to outbid bidder \( j \) and take the \( j \)th slot. \( ^{1} \) Or \( \varphi_i^{-1}(0) \) if \( \varphi_j(v_j) < 0 \).

The payment is computed as follows:

\[
p_i(v_i, v_{-i}) = v_i x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(z, v_{-i}) \, dz,
\]

\[
 = v_i \cdot a_i - \left[ \int_0^{b_k} 0 \, dz + \int_{b_k}^{b_{k-1}} a_k \, dz + \int_{b_{k-1}}^{b_{k-2}} a_{k-1} \, dz + \cdots + \int_{b_{i+1}}^{b_i} a_{i+1} \, dz + \int_{b_i}^{v_i} a_i \, dz \right],
\]

\[
 = v_i \cdot a_i - \left[ (b_{k-1} - b_k)a_k + (b_{k-2} - b_{k-1})a_{k-1} + \cdots + (b_i - b_{i+1})a_{i+1} + (v_i - b_i)a_i \right],
\]

\[
 = b_i a_i - \left[ (b_{k-1} - b_k)a_k + (b_{k-2} - b_{k-1})a_{k-1} + \cdots + (b_i - b_{i+1})a_{i+1} \right],
\]

\[
 = b_i a_i - \sum_{l=k}^{i+1} (b_{l-1} - b_l)a_l.
\]