Bayes-Nash Equilibrium in the First-Price Auction

We state and prove a Bayes-Nash Equilibrium strategy for the first-price auction, assuming valuations are drawn i.i.d. from the uniform distribution on [0, 1].

**Theorem 0.1.** In a first-price auction in which agents’ valuations are drawn i.i.d. from the uniform distribution on [0, 1], the following bid by each bidder $i$ is a Bayes-Nash equilibrium:

$$b_i(v_i) = \left(\frac{n-1}{n}\right) v_i$$

**Proof.** Fix a bidder $i$. We assume that all bidders besides $i$ bid according to this formula, then argue that $i$ should do the same.

Let $v'$ denote the highest bid among the $n-1$ bidders other than $i$. Let $z$ represent $i$’s bid. There are two possible cases:

- $z \leq \left(\frac{n-1}{n}\right) v'$. Here, $i$ does not win the item, so $u_i = 0$. Based on the result above, $P\left(z \leq \frac{n-1}{n} v'\right) = 1 - \left(\frac{nz}{n-1}\right)^{n-1}$.

- $z \geq \left(\frac{n-1}{n}\right) v'$. In this case, $i$ wins the item, receiving utility $u_i = v_i - z$. The probability of this case occurring is computed as follows:

$$P\left(z \geq \frac{n-1}{n} v'\right) = P\left(v' \leq \frac{nz}{n-1}\right)$$

$$= F_{v'}\left(\frac{nz}{n-1}\right)$$

$$= F_{X \sim \text{Unif}[0,1]}\left(\frac{nz}{n-1}\right)^{n-1}$$

$$= \begin{cases} 
\left(\frac{nz}{n-1}\right)^{n-1} & \text{if } z \leq \frac{n-1}{n} \\
1 & \text{otherwise},
\end{cases}$$

To comment on the last equality, if $z > \frac{n-1}{n}$ then $\left(\frac{nz}{n-1}\right)^{n-1} > 1$, in which case the cdf is just 1. However, if a bid of $\frac{n-1}{n}$ will guarantee that $i$ wins the auction, there is no point in bidding beyond that. Therefore, we restrict the value of $z$ to $[0, \frac{n-1}{n}]$.

When bidding $z \in [0, \frac{n-1}{n}]$, bidder $i$’s expected utility is given by the following:

$$E[u_i] = \left(\frac{nz}{n-1}\right)^{n-1} (v_i - z) + \left(1 - \left(\frac{nz}{n-1}\right)^{n-1}\right) \cdot 0$$
\[ = \left( \frac{n}{n-1} \right)^{n-1} z^{n-1}(v_i - z). \]

Now, we take the derivative with respect to \( z \) to maximize \( i \)'s expected utility.

\[
\frac{dE[u_i]}{dz} = \frac{d}{dz} \left[ \left( \frac{n}{n-1} \right)^{n-1} z^{n-1}(v_i - z) \right] = 0
\]

\[
\frac{d}{dz} \left[ z^{n-1}(v_i - z) \right] = 0
\]

\[
(n - 1)z^{n-2}(v_i - z) - z^{n-1} = 0
\]

\[
(n - 1)(v_i - z) - z = 0
\]

\[
(n - 1)v_i - nz = 0
\]

\[
z = \frac{n - 1}{n} v_i.
\]

Bidder \( i \) maximizes her utility by bidding \( \left( \frac{n - 1}{n} \right) v_i \). To be sure of this, we verify that neither of the two extreme points \( z = 0 \) and \( z = \frac{n - 1}{n} \) can give positive utility, and that the second derivative of \( E[u_i](z) \) is positive at \( z = \left( \frac{n - 1}{n} \right) v_i. \) \( \Box \)