Introduction to Posted-Price Mechanisms
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We introduce the posted-price mechanism, and then discuss revenue maximization in this simple model.

1 The Posted-Price Mechanism

In the space of designs by which to auction off a single good, we have seen that there is a significant difference between a payment scheme that charges the second-highest bid, and one in which bidders pay their bids. In this lecture, we introduce a new payment and allocation scheme, called the posted-price mechanism, in which the auctioneer announces (i.e., posts) the price $\pi$ at which they are selling the good,\(^1\) after which any bidder that bids at least the posted price is uniformly eligible to win the good. The winner is then charged the posted price $\pi$, and all other bidders pay nothing.

Given a posted price mechanism with price $\pi$, we can ask:

- How should bidders behave: i.e., what should they bid?
- Assuming they behave as predicted, how good is this outcome?

“Good” in the second question implies that we are measuring something. Today, it will be total expected revenue, which is the sum of the total expected payments (equivalently, expected total payments):

$$\sum_{i \in N} \mathbb{E}_{v \sim F} [p_i(b_i, b_{-i})] = \mathbb{E}_{v \sim F} \left[ \sum_{j \in N} p_i(b_j, b_{-j}) \right].$$

2 Dominant Strategies

The analysis required to determine what strategy bidders should use is similar to that of the second-price, sealed-bid auction. The (familiar) case analysis is described graphically in Figure 1 and Figure 2, and summarized in Figure 3. Thus, we see that the posted-price mechanism for one good is DSIC: regardless of what any other bidder does, bidding one’s true value is a dominant strategy.

Now, observe that because payments for the winner are predetermined by the auctioneer, a bid in this mechanism is in fact a binary signal. Placing a bid $b_i \geq \pi$ is telling the auctioneer “I am willing to purchase the good at that price.” Placing a bid $b_i < \pi$ is telling the auctioneer “I am not willing to purchase the good at that

\(^1\) Sort of like how when we go out for coffee, we pay whatever price the establishment has decided on, irrespective of possible competition from other people. (Well, aside from the long lines.)
price." Thus, the following bids also comprise a dominant strategy:

\[ b_i \in \begin{cases} 
[\pi, \infty), & \text{if } v_i \geq \pi \\
(-\infty, \pi), & \text{otherwise.} 
\end{cases} \]

This means that there are multiple dominant strategies in this mechanism. For example, the following two strategies would each do just as well as bidding truthfully in a posted-price mechanism:

\[ b_i = \begin{cases} 
\pi, & \text{if } v_i \geq \pi \\
0, & \text{otherwise} 
\end{cases} \]

\[ b_i = \begin{cases} 
\infty, & \text{if } v_i \geq \pi \\
-\infty, & \text{otherwise.} 
\end{cases} \]

Because of the myriad of dominant strategies, the posted-price mechanism differs from the second-price auction in an important way. *Bidders need not bid truthfully!* In particular, bidders need not divulge their private information in order to maximize their utility.
Revenue Maximization

Unlike the basic first- and second-price auctions (without a reserve), which always produce a winner, the posted-price mechanism has no such guarantee. For example, if the posted price is larger than the upper bound on bidder values, then no one will ever win. Thus, using distributional knowledge about bidder types, the auctioneer should reason about what an appropriate posted price would be to, for example, maximize total expected revenue.

One observation we can make right off the bat is that the posted-price mechanism can sometimes generate more revenue than the second-price auction. For a given value profile \( v \), the second-price auction accrues revenue \( v(n-1) \), but posting a price of \( \pi \) for the same profile, with \( v(n-1) < \pi \leq v(n) \), would generate \( \pi - v(n-1) \) more revenue than the second-price auction. On the other hand, posting a price of \( \pi < v(n-1) \) could also generate less revenue, depending on how winners are determined (i.e., how ties are broken).

Solving for the expected revenue-maximizing posted price is straightforward in the case of only one bidder. If the mechanism posts price \( \pi \), then the probability of a sale is equal to the probability that the bidder’s value is at least \( \pi \): \( \Pr(v \geq \pi) = 1 - \Pr(v \leq \pi) = 1 - F(\pi) \). Hence, the expected revenue is given by:

\[
R(\pi) = \pi \left(1 - F(\pi)\right).
\]

More concretely, when values are distributed uniformly on \([0,1]\), then the expected revenue at posted price \( \pi \) is given by:

\[
R(\pi) = \pi \left(1 - \pi\right).
\]

This revenue curve is depicted in Figure 4.

Why does the revenue curve have this shape? Well, if the posted price is 0, then revenue is necessarily 0. Moreover, increasing the
posted price by a little bit increases revenue by a little bit. At the other extreme, if the posted price is 1, then revenue is again 0, as the probability of drawing a bidder whose type is 1 is 0. Again, decreasing the posted price by a little bit increases revenue by a little bit.

To maximize expected revenue, we take the derivative of the revenue curve with respect to \( \pi \):

\[
[R(\pi)]' = [\pi (1 - F(\pi))]'
= [\pi] [1 - F(\pi)]' + [\pi]' [1 - F(\pi)]
= -\pi f(\pi) + 1 - F(\pi)
= \pi - \frac{1 - F(\pi)}{f(\pi)},
\]

where we divide by \(-f(\pi)\) in the last step. Do you recognize the form of this final function? It is a virtual value function! Hence, the derivative of the revenue curve is the virtual value function.

At this point, the first-order conditions tell us to solve for the value \( \pi^* \) such that \( \varphi(\pi^*) = 0 \), which we can do by performing a function inverse: \( \pi^* = \varphi^{-1}(0) \). This value is called the monopoly (reserve) price, and revenue is maximized by posting this price.

For example, if \( F(\pi) \) is uniform on \([0, 1]\), so that \( F(\pi) = \pi \) and \( f(\pi) = 1 \), then the virtual value function is:

\[
\varphi(\pi) = \pi - \frac{1 - F(\pi)}{f(\pi)}
= \pi - \frac{1 - \pi}{1}
= 2\pi - 1.
\]

This function is 0, when \( \pi = 1/2 \). Eyeballing Figure 4, we confirm that indeed \( \pi = 1/2 \) is the point at which revenue is maximized. Therefore, assuming a single bidder whose value is drawn uniformly from \([0, 1]\), the monopoly price, i.e., the posted price \( \pi^* \) at which \( \varphi(\pi^*) = 0 \), and hence revenue is maximized, is \( \pi^* = 1/2 \).

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\(^2\) This analysis relies on the assumption that \( f(\pi) > 0 \) everywhere.

\(^3\) This might arouse suspicion, as there may be multiple values \( \pi^* \) for which \( \varphi(\pi^*) = 0 \). For now, simply assume virtual values are strictly increasing, so that this point is necessarily unique.