We present a method for choosing a posted price in a symmetric setting, along with a guarantee about the expected welfare of the method. Once again, this analysis demonstrates the tradeoff between optimal, complex mechanisms and approximately-optimal, simple mechanisms.

1 Expected Welfare

Suppose a set of \( n \) symmetric bidders, with values drawn i.i.d. from some distribution \( F \). The expected welfare when using posted price \( \pi \) is the probability the good is allocated at price \( \pi \) (i.e., the probability of a sale) times the welfare at price \( \pi \):\(^1\)

\[
W(\pi) = \Pr(\text{good is allocated at } \pi) \cdot \mathbb{E}_{v \sim F}[v \mid v \geq \pi].
\]

Since the probability that each bidder’s value is at most \( \pi \) is \( F(\pi) \), the probability of a sale is:

\[
\Pr(\text{good is allocated at } \pi) = \Pr(\text{someone’s value is above } \pi) = 1 - \Pr(\text{good is not allocated at } \pi) = 1 - \Pr(\text{no one’s value is above } \pi) = 1 - \Pr(\text{everyone’s value is below } \pi) = 1 - F(\pi)^n.
\]

Now, the welfare at price \( \pi \) is:

\[
\mathbb{E}_{v \sim F}[v \mid v \geq \pi] = \frac{\mathbb{E}_{v \sim F}[v \textbf{1}_{v \geq \pi}]}{\Pr(v \geq \pi)} = \frac{\int_{\pi}^{\infty} vf(v) \, dv}{1 - F(\pi)}.
\]

Therefore, the expected welfare at posted price \( \pi \) is:

\[
W(\pi) = (1 - F(\pi)^n) \left( \frac{\int_{\pi}^{\infty} vf(v) \, dv}{1 - F(\pi)} \right).
\]

To maximize \( W(\pi) \), we could compute the first derivative, and then find the posted price at which this derivative equals 0. As it turns out, this isn’t any fun (Read: it does not yield an easy-to-interpret solution). Consequently, we take an alternative approach in this lecture. We analyze the expected welfare not of the optimal posted price, but of an approximately-optimal posted price instead, and one that corresponds to a relatively simple mechanism.
A Simple Posted-Price Mechanism

One reasonable (as we will show) way to choose a posted price is so that, in expectation, there will be one winner. In a symmetric setting, this amounts to setting $\pi$ such that the probability of drawing a value $v \geq \pi$ is $1/n$. Then, in expectation, there will be one winner as

$$E\left[\sum_{i \in N} 1_{v_i \geq \pi}\right] = \sum_{i \in N} E[1_{v_i \geq \pi}] = \sum_{i \in N} \Pr(v_i \geq \pi) = \sum_{i \in N} \frac{1}{n} = 1.$$ 

The posted price $\pi^*$ for which $\Pr(v \geq \pi) = 1/n$ is $F^{-1}(1 - 1/n)$, since $F(\pi) = \Pr(v < \pi) = 1 - 1/n$. In this lecture, we are interested in bounding the expected welfare of the mechanism that posts price $\pi^*$. We call this expected welfare APX.

If for each bidder $i$,

$$\Pr(\pi \leq v_i) = 1/n,$$

then for each bidder $i$,

$$\Pr(v_i < \pi) = 1 - 1/n.$$ 

Moreover,

$$\Pr(\forall i, v_i < \pi) = (1 - 1/n)^n,$$

and

$$\Pr(\exists i, v_i < \pi) = 1 - (1 - 1/n)^n.$$ 

In other words,

$$\Pr(\text{good is allocated at } \pi) = 1 - (1 - 1/n)^n,$$

and, by symmetry,

$$\text{APX} = \Pr(\text{good is allocated at } \pi) \ E_{v \sim F} [v \mid v \geq \pi]$$

$$= \left(1 - (1 - 1/n)^n\right) \ E_{v \sim F} [v \mid v \geq \pi].$$

The optimal welfare (OPT) in this setting can be achieved via a second-price auction. But we will obtain a looser upper bound on OPT, which relates to the lower bound we will derive on APX.

A Posted-Price Mechanism with $n$ Identical Goods

We construct an upper bound on OPT by analyzing (via a series of examples) the total expected welfare of the second-price auction as compared to that of a posted-price mechanism with $n$ identical goods. Assuming $n$ goods, everyone can be served.

There are instances (i.e., value profiles of dimension $n$ such that $v_{(n)} < \pi$) where a posted-price mechanism (assuming only one good) does not sell the good, while a second-price auction would.
But we claim that \( n \) posted-price mechanisms will always obtain more welfare, in expectation, than a single second-price auction. There are some cases where the posted-price mechanism does not sell the good, but there are others where the good is sold to multiple bidders. As it turns out, the latter always makes up for the former.

We do not prove this upper bound formally. Instead, we present a few (hopefully) insightful examples. In all our examples, there are \( n = 2 \) symmetric bidders, each of whom draws their value from a discrete distribution \( F \), with two possible types. The low type is drawn with probability \( q \), and the high type is drawn with probability \( 1 - q \).

**Example 3.1.** Suppose there are two types, \( T = \{0, 1\} \), each drawn with equal probability: i.e., \( q = 1 - q \). There are four possible bidder profiles, each of which occur with equal probability:

\[
v = \begin{cases} 
(0,0), & \Pr(v) = 0.25 \\
(0,1), & \Pr(v) = 0.25 \\
(1,0), & \Pr(v) = 0.25 \\
(1,1), & \Pr(v) = 0.25.
\end{cases}
\]

- The expected welfare of a second-price auction will generate is \( \frac{3}{4} \).
  \[
  \frac{1}{4}(0 + 1 + 1 + 1) = \frac{3}{4}.
  \]
- The expected welfare of a posted-price mechanism with one good and posted price \( \pi \in (0, 1] \) is \( \frac{3}{4} \).
  \[
  \frac{1}{4}(0 + 1 + 1 + 1) = \frac{3}{4}.
  \]
- The expected welfare of a posted-price mechanism with \( n \) copies of the good and posted price \( \pi \in (0, 1] \) is \( 1 \).
  \[
  \frac{1}{4}(0 + 1 + 1 + 2) = 1.
  \]

This example shows that the total expected welfare of a posted-price mechanism with \( n \) goods can be strictly greater than the expected welfare of a second-price auction. We now show that these values can be made to be arbitrarily close.

**Example 3.2.** Suppose there are two types, \( T = \{1 - 4\epsilon, 1\} \), for some \( \epsilon \in (0, 1] \), each drawn with equal probability: i.e., \( q = 1 - q \). There are four possible bidder profiles, each of which occur with equal probability:

\[
v = \begin{cases} 
(1 - 4\epsilon, 1 - 4\epsilon), & \Pr(v) = 0.25 \\
(1 - 4\epsilon, 1), & \Pr(v) = 0.25 \\
(1, 1 - 4\epsilon), & \Pr(v) = 0.25 \\
(1, 1), & \Pr(v) = 0.25.
\end{cases}
\]
• The expected welfare of a second-price auction is
\[
\frac{1}{4} (1 - 4\epsilon + 1 + 1 + 1) = \frac{1}{4} (4 - 4\epsilon) = 1 - \epsilon.
\]

• The expected welfare of a posted-price mechanism with one good and posted price \( \pi > 1 - 4\epsilon \) is \( \frac{3}{4} \).
\[
\frac{1}{4} (0 + 1 + 1 + 1) = \frac{3}{4}.
\]

• The expected welfare of a posted-price mechanism with \( n \) copies of the good and posted price \( \pi > 1 - 4\epsilon \) is \( \frac{3}{4} \).
\[
\frac{1}{4} (0 + 1 + 1 + 2) = 1.
\]

Whereas the performance of the posted-price mechanism with one good matched that of the second-price auction in the first example, in this example, for small values of \( \epsilon \), the second-price auction generates strictly more expected welfare than the posted-price mechanism with one good. On the other hand, a posted-price mechanism with \( n \) copies of the good outperforms the second-price auction, for \( \epsilon > 0 \).

In both our examples so far, \( n \) posted-price mechanisms have outperformed the second-price auction (though not by much in the second example). How might the second-price auction outperform \( n \) posted-price mechanisms? The only way would be to shift mass to lower types which would not be allocated in the \( n \) posted-price mechanisms, but would be in the second-price auction. However, when mass shifts to lower types, so, too, does the requisite posted price, which in our examples is \( F^{-1} (1/2) \). As a result, the second-price auction never strictly outperforms \( n \) posted-price auctions in our setting. Our final example demonstrates this result, assuming a high probability of low-type profiles.

**Example 3.3.** Suppose there are two types, \( T = \{1 - \epsilon, 1\} \), for some \( \epsilon \in (0, 1] \), and with probability \( q \gg 1 - q \), a bidder has type \( 1 - \epsilon \). As usual, there are four possible bidder profiles, but now the profile in which both bidders have low types is significantly more probable:

\[
\mathbf{v} = \begin{cases}
(1 - \epsilon, 1 - \epsilon), & \Pr(\mathbf{v}) = q^2 \\
(1 - \epsilon, 1), & \Pr(\mathbf{v}) = q(1 - q) \\
(1, 1 - \epsilon), & \Pr(\mathbf{v}) = (1 - q)q \\
(1, 1), & \Pr(\mathbf{v}) = (1 - q)^2.
\end{cases}
\]

• The expected welfare of a second-price auction is
\[
\left[q^2 \right] (1 - \epsilon) + 2 \left[q(1 - q)\right] (1) + \left[(1 - q)^2\right] (1).
\]
• The expected welfare of a posted-price mechanism with one good and posted price be \( \pi \in (0, 1 - \epsilon] \) is
\[
q^2 (1 - \epsilon) + 2[q(1 - q)] \left( \frac{(1 - \epsilon) + 1}{2} \right) + \left[ (1 - q)^2 \right](1).
\]

• The expected welfare of a posted-price mechanism with \( n \) copies of a good and posted price be \( \pi \in (0, 1 - \epsilon] \) is
\[
q^2 (1 - \epsilon)2 + 2[q(1 - q)](2 - \epsilon) + \left[ (1 - q)^2 \right]2.
\]
Comparing the terms for the \( n \) posted-price mechanisms with the terms for the second-price auction, we see that each term in former is at least that of the corresponding term in the latter.

Based on this series of examples, we make the following claim (without proof):

**Proposition 3.4.** The optimal welfare (i.e., the welfare of the second-price auction) is upper-bounded by the welfare of a posted-price mechanism that can sell up to \( n \) identical copies of the good.

4 Bounds and an Approximation Ratio

We are now ready to derive an upper bound on \( \text{OPT} \) and a lower bound on \( \text{APX} \), as well as an approximation ratio for the total expected welfare of our simple posted-price mechanism, assuming the bidders’ values are drawn i.i.d..

**Lemma 4.1.** The optimal welfare is upper-bounded as follows:
\[
\text{OPT} \leq \mathbb{E}[v \mid v \geq \pi].
\]

**Proof.** By Proposition 3.4, it suffices to calculate the expected welfare of a posted-price mechanism that can sell up to \( n \) copies of the good.
Since bidders do not compete against each other in this mechanism, we can consider each bidder’s impact on welfare independently of any others’. Hence, the expected welfare of such a mechanism is:
\[
\text{OPT} \leq \sum_{i=1}^{n} \Pr(v_i \geq \pi) \mathbb{E}[v_i \mid v_i \geq \pi]
= \sum_{i=1}^{n} \frac{1}{n} \mathbb{E}[v \mid v \geq \pi]
= \frac{1}{n} \mathbb{E}[v \mid v \geq \pi]
\]
The middle equality follows from the fact that \( \Pr(v_i \geq \pi) = 1/n \) and that all bidders are symmetric.
Lemma 4.2. The expected welfare of our simple posted-price mechanism, when all values are drawn i.i.d., is lower bounded as follows:

\[
APX \geq (1 - 1/e) \mathbb{E}[v \mid v \geq \pi].
\]

Proof. Recall that the expected welfare generated by the simple posted-price mechanism is:

\[
APX = (1 - (1 - 1/n)^n) \mathbb{E}[v \mid v \geq \pi].
\]

To simplify this expression, we bound the term \((1 - 1/n)^n\).

Let \(f(n) = (1 - 1/n)^n\). In the limit, as \(n\) tends toward infinity,\(^2\)

\[
\lim_{n \to \infty} f(n) = 1/e.
\]

Moreover, \(f(n)\) is an increasing function when \(n > 1\), as shown in Figure 1. It follows that

\[
(1 - 1/n)^n \leq 1/e,
\]

or equivalently,

\[
1 - (1 - 1/n)^n \geq 1 - 1/e.
\]

Therefore,

\[
APX = (1 - (1 - 1/n)^n) \mathbb{E}[v \mid v \geq \pi]
\geq (1 - 1/e) \mathbb{E}[v \mid v \geq \pi].
\]

Figure 1: A comparison between \(f(n) = (1 - 1/n)^n\) and \(1/e\). Observe that \(f(n)\) is an increasing function when \(n \geq 1\). Moreover, it approaches \(1/e\) from below.

Now that we have both an upper bound on OPT, and a lower bound on APX, we can derive an approximation ratio.
**Theorem 4.3.** The approximation ratio of the welfare generated by our simple posted-price mechanism, when all values drawn i.i.d., is

\[
\frac{\text{APX}}{\text{OPT}} \geq 1 - \frac{1}{e}.
\]

**Proof.** Start with the lower bound:

\[
\text{APX} \geq (1 - 1/e) \mathbb{E}[v | v \geq \pi].
\]

Divide by OPT:

\[
\frac{\text{APX}}{\text{OPT}} \geq \frac{(1 - 1/e) \mathbb{E}[v | v \geq \pi]}{\mathbb{E}[v | v \geq \pi]} = 1 - \frac{1}{e}.
\]

This result is surprising. What it means is that by posting a price of \( F^{-1}(1 - 1/n) \), we can obtain at least 0.63 of the optimal expected welfare. One way to interpret this result is that competition, which is downplayed by this mechanism but is a key ingredient of auctions, accounts for 0.37 of the optimal expected welfare.

**A. The Exponential Function**

Let

\[
f(x, n) = (1 + x/n)^n.
\]

We investigate this function in the limit, as \( n \) approaches \( \infty \), by taking logs:

\[
\log \left( \lim_{n \to \infty} (1 + x/n)^n \right) = \lim_{n \to \infty} n \log (1 + x/n) = \lim_{n \to \infty} \log \left( \frac{1 + x/n}{1/n} \right).
\]

BY L'Hôpital's rule,

\[
\lim_{n \to \infty} \log \left( \frac{1 + x/n}{1/n} \right) = \lim_{n \to \infty} \frac{\log (1 + x/n)'}{(1/n)'} = \lim_{n \to \infty} \frac{-x/n^2}{(1 + x/n)(-1/n^2)}.
\]
\[
\lim_{n \to \infty} \frac{x}{1 + x/n} = x.
\]

Since
\[
\log \left( \lim_{n \to \infty} (1 + x/n)^n \right) = x,
\]
it follows that
\[
\lim_{n \to \infty} (1 + x/n)^n = e^x.
\]

Letting \( x = -1 \) yields
\[
\lim_{n \to \infty} (1 - 1/n)^n = e^{-1}.
\]

By taking logs, we restricted the domain of \( f \) to \( n > 1 \), because when \( n = 1 \), \( \log (1 - 1/n) = -\infty \). For \( n > 1 \), \( \log (1 - 1/n) \) is an increasing function, and so is \( n \log (1 - 1/n) = \log f(-1,n) \). Finally, since \( \log \) itself is increasing, \( f(-1,n) \) is also increasing. Therefore,
\[
(1 - 1/n)^n \leq 1/e.
\]