Approximately Optimal Auctions
Professor Greenwald
2017-02-21

We show that simple auctions can generate near-optimal revenue by using non-optimal reserve prices.

1 Simple vs. Optimal Auctions

While Myerson’s theorem provides an elegant solution to the optimal auction design problem for \( k \)-good auctions, his auction is rather complicated, and is consequently not widely used in practice. On the other hand, posted-price mechanisms are widely used in practice. We have derived an optimal posted-price mechanism for an auction with a single bidder.\(^1\) What about for multiple bidders? What is the optimal posted-price then? We can answer this question by taking derivatives, etc.,\(^2\) but the answer will ultimately depend on distributional assumptions. We will take an alternative approach in this lecture, analyzing not the optimal mechanism, but a randomized posted-price mechanism instead. Perhaps surprisingly, the guarantees we derive will be independent of any distributional assumptions.

The mechanism we will analyze is “posting a random price”. We will analyze this mechanism in some detail, first assuming only one bidder. We will then generalize to \( n \) symmetric bidders’, where all bidders’ values are drawn from the same distribution, assuming digital goods, or equivalently, goods in infinite supply.

2 The Posted-Price Mechanism, Revisited

Recall the setup of a posted-price mechanism with only one bidder, whose value is drawn from distribution \( F \). The expected revenue is:

\[
R(\pi) = \pi(1 - F(\pi)).
\]

Maximizing expected revenue is akin to selecting a posted price \( \pi \) s.t.

\[
\pi^* \in \arg \max_{\pi \in T_i} R(\pi).
\]

The posted price that maximizes revenue is called the monopoly price, or the monopoly reserve, and is equal to \( \pi^* = \varphi^{-1}(0) \). The corresponding optimal revenue \( \text{OPT} = R(\pi^*) \).

Now, suppose that rather than setting the price at the monopoly reserve, the seller selects a price at random, simply by sampling from the distribution \( F \). Call the expected revenue of this mechanism \( \text{APX} \):

\[
\text{APX} = \mathbb{E}[R(\pi)]
\]
= \int_{v}^{\pi} \pi (1 - F(\pi)) f(\pi) \, d\pi

Just how well can this mechanism do? That is, what is the ratio of APX to OPT? We will develop some machinery in this lecture that will enable us to answer this question. The machinery is a bit complex, but with it, the answer to the question will be simple.

Here is a summary of the necessary machinery, together with our plan for the lecture:

1. We introduce quantiles, and review the probability integral transform and inverse transform sampling.

2. We redefine revenue in terms of quantiles, which leads to a simple interpretation of APX as the area under the revenue curve.

3. Next, we show that the virtual value function is the derivative of the revenue curve. By the regularity assumption, this derivative is non-decreasing, which implies the revenue curve is concave.

4. Finally, we derive an approximation ratio by picture.

5. All of the above applies only in the single-bidder case. We conclude by showing how to extend this reasoning to a symmetric setting with multiple bidders, assuming infinite supply. The final result is an approximation ratio for a prior-independent posted-price mechanism with multiple bidders, assuming infinite supply.

3 Reasoning in Probability Space

A quantile \( q \in [0, 1] \) is the relative strength of a value \( v \in T \):

\[ q(v) = 1 - F(v). \]

Likewise,

\[ F(v) = 1 - q(v). \]

The value \( F(v) \) is the probability that the value of a random draw from the distribution \( F \) is less than or equal to \( v \). Accordingly, the quantile \( v \) is the probability that the value of a random draw is greater than \( v \). Thus, lower quantiles correspond to higher types/values, and higher quantiles correspond to lower types/values.

To map a quantile back to a value, we invert \( F \) at \( 1 - q \):

\[ v(q) = F^{-1} (1 - q). \]

By the probability integral transform (defined presently), selecting a quantile uniformly at random is akin to sampling a value from \( F \).
3.1 Probability Integral Transform

We give a brief explanation of the probability integral transform. Let $X$ be a random variable with arbitrary CDF $F$. Now define the random variable $Y = F(x)$. Note that the range of $Y$ is $[0, 1]$. One interesting question is, how is $Y$ distributed? That is, what is the CDF of an arbitrary CDF? The answer is straightforward to derive, but surprising nonetheless: for $0 \leq y \leq 1$,

\[
\Pr(Y \leq y) = \Pr(F(x) \leq y) = \Pr(x \leq F^{-1}(y)) = F(F^{-1}(y)) = y.
\]

What this says in words is that $Y$ is uniformly distributed on $[0, 1]$! That is, the CDF of an arbitrary CDF $F$ is uniform. Finally, since a quantile associated with a value is 1 less the CDF at that value, quantiles are likewise uniformly distributed.

Remark 3.1. The inverse of the probability integral transform is the following: if $U$ is a uniform random variable on $[0, 1]$, then the random variable $F^{-1}(U)$ is distributed according to $F$: for $0 \leq u \leq 1$,

\[
\Pr(F^{-1}(u) \leq x) = \Pr(u \leq F(x)) = F(x).
\]

An important practical consequence of this observation is a process for sampling from an arbitrary CDF: first sample from the uniform distribution to obtain a value of $F(x)$, and then apply $F^{-1}$ to that sample to recover $x = F^{-1}(F(x))$, which is necessarily distributed according to $F$. This process is called inverse transform sampling.

Example 3.2. The exponential distribution has CDF $F(x) = 1 - e^{-\lambda x}$. We invert this CDF as follows:

\[
F(x) = 1 - e^{-\lambda x} \\
e^{-\lambda x} = 1 - F(x) \\
-\lambda x = \ln(1 - F(x)) \\
x = -\frac{\ln(1 - F(x))}{\lambda}.
\]

Now, using inverse transform sampling, we can sample from the exponential distribution by first sampling a value $u = F(x)$ from $U[0, 1]$, and then plugging the sampled value $u$ into the function $-\ln(1 - u)/\lambda$. Equivalently, we can sample a quantile $q = 1 - u$ from $U[0, 1]$, and then plug $q$ into the function $-\ln(q)/\lambda$. 
3.2 Revenue Curves in Quantile Space

With our newfound knowledge of quantiles in mind, let’s revisit our definition of the revenue curve. Recall the formula for expected revenue at posted price $\pi$:

$$R(\pi) = \pi(1 - F(\pi)).$$

In words, this quantity is the product of the posted price and the probability of a sale. The probability of a sale is the probability that a draw from $F$ exceeds $\pi$. But this is exactly what a quantile represents! So at a given quantile $q$, the probability of a sale is simply $q$. Moreover, the value of a sale at quantile $q$ is $v(q) = F^{-1}(1 - q)$: i.e.,

$$R(q) = F^{-1}(1 - q)q,$$

Next, let’s investigate expected revenue in quantile space:

$$\mathbb{E}[R(q)] = \mathbb{E}[F^{-1}(1 - q)q]$$

$$= \int_0^1 F^{-1}(1 - q)q f(q) \, dq$$

$$= \int_0^1 F^{-1}(1 - q)q \, dq$$

The first step in this derivation follows from the definition of the revenue curve. The second step follows from the definition of expectation. The third step follows from the fact that quantiles are necessarily uniformly distributed, so that $f(q) = 1$. In words, in quantile space, the expected revenue is the area under the revenue curve.

Now observe the following:

$$\frac{dq(\pi)}{d\pi} = \frac{d}{d\pi} (1 - F(\pi))$$

$$= -f(\pi).$$

Equivalently, $dq(\pi) = -f(\pi) \, d\pi$. Therefore,

$$\mathbb{E}[R(q(\pi))] = \int_0^1 F^{-1}(1 - q(\pi))q(\pi) \, dq(\pi)$$

$$= - \int_0^1 F^{-1}(1 - q(\pi))q(\pi) \, dq(\pi)$$

$$= \int_0^\pi \pi(1 - F(\pi))f(\pi) \, d\pi$$

$$= \text{APX}$$

In sum, we have expressed APX in quantile space as expected revenue. It follows that APX is the area under the revenue curve.

Sample revenue curves are plotted in quantile space in Figures 1 and 2. The area under these curves is the expected revenue.
Figure 1: Revenue curve corresponding to the exponential distribution, $\lambda = 1$, plotted in quantile space.

Figure 2: Revenue curve of the uniform distribution, plotted in quantile space.
4 Properties of the Revenue Curve

We have shown that APX is the area under the revenue curve in quantile space. But if the revenue curve is arbitrarily complex, it may be difficult to compute this integral. We now set out to show that the revenue curve cannot be arbitrarily complex; on the contrary, it is always concave, assuming $F$ is regular.

4.1 Virtual Values

For starters, we show how virtual values relate to the revenue curve. Specifically, we differentiate the revenue curve $R$ w.r.t. quantile $q$:

$$\frac{dR(q)}{dq} = \frac{dq}{dq} F^{-1}(1 - q)$$

$$= [q]'[F^{-1}(1 - q)] + [q][F^{-1}(1 - q)]'.

To differentiate the function inverse, we use the chain rule. For a function $f(g(x))$, let $z = f(y)$ and $y = g(x)$. Then:

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = \frac{df(y)}{dy} \frac{dg(x)}{dx}.

Notice that $x = F(F^{-1}(x))$. Taking the derivatives of both sides of this equation, letting $z = F(y)$ and $y = F^{-1}(x)$, yields

$$1 = \frac{dz}{dx} = \frac{dF(y)}{dy} \frac{dF^{-1}(x)}{dx} = f(y)[F^{-1}(x)]'.

Rearranging,

$$[F^{-1}(x)]' = \frac{1}{f(y)} = \frac{1}{f(F^{-1}(x))}.

Thus,

$$[q][F^{-1}(1 - q)]' = \frac{-q}{f(F^{-1}(1 - q))}.

Now, continuing where we left off,

$$\frac{dR(q)}{dq} = F^{-1}(1 - q) \left( \frac{-q}{f(F^{-1}(1 - q))} \right).

Since $q(v) = 1 - F(v)$ and $v = F^{-1}(1 - q(v))$, we conclude that

$$\frac{dR(q)}{dq} = v - \frac{1 - F(v)}{f(v)} = \varphi(v).

Therefore, the derivative of the revenue curve, also called the marginal revenue, is the virtual value function!
4.2 Concave Revenue Curves

We now set out to prove that the revenue curve is concave. To do so, we need to assume the virtual value function (in value space) is non-decreasing, or equivalently, the virtual value function (in quantile space) is non-increasing.

As an example, if values are uniformly distributed on $[0, 1]$, the virtual value function (in value space) $\phi(v) = 2v - 1$ is non-decreasing. Since $v(q) = F^{-1}(1 - q) = 1 - q$, the virtual value function in quantile space $\phi(q) = 1 - 2q$ is non-increasing.

**Definition 4.1 (Concave function).** A function $f$ is concave if, for any $c \in [0, 1]$,

$$f((1-c)x + cy) \geq (1-c)f(x) + cf(y).$$

Equivalently, for an $x, y$ in the domain,

$$f \left( \frac{x+y}{2} \right) \geq \frac{f(x) + f(y)}{2}.$$

**Remark 4.2.** You can understand concavity by graphing $f$. Draw a line from point $(x, f(x))$ to $(y, f(y))$. The function $f$ is concave if it lies above the line in the interval $[x, y]$, for all choices of $x$ and $y$.

**Proposition 4.3.** Assuming regularity, the revenue curve is concave.

**Proof.** We show that the revenue curve must be concave using a bit of calculus. Let $q_1 \leq q_2$, so that $v(q_1) \geq v(q_2)$. Integrating the virtual value function from quantile $q_1$ to $q_2$ yields:

$$\int_{q_1}^{q_2} \phi(v(q)) \, dq = \int_{q_1}^{q_2} R(q) \, dq = R(q_2) - R(q_1).$$

It follows that

$$\int_{q_1}^{q_2} \phi(v(q)) \, dq = R \left( \frac{q_1 + q_2}{2} \right) - R \left( q_1 \right)$$

and

$$\int_{\frac{q_1 + q_2}{2}}^{q_2} \phi(v(q)) \, dq = R \left( q_2 \right) - R \left( q_1 + \frac{q_2}{2} \right).$$

Since the virtual value function is non-increasing in quantile space,

$$\int_{q_1}^{\frac{q_1 + q_2}{2}} \phi(v(q)) \, dq \geq \int_{\frac{q_1 + q_2}{2}}^{q_2} \phi(v(q)) \, dq \geq R \left( q_2 \right) - R \left( \frac{q_1 + q_2}{2} \right) \geq \frac{R(q_2) - R(q_1)}{2}.$$

We conclude that the revenue curve is concave. \qed
We can also proceed via proof by picture to show that integrating a non-increasing function yields a concave function.

**Theorem 4.4.** Let \( f \) be a positive, real-valued differentiable function defined for all \( x \geq a \). Consider a function \( F \) defined by \( F(x) = \int_a^x f(t)dt \). If \( f \) is non-increasing on interval \([a, b]\), then \( F \) is concave on that interval.

**Proof by picture.** Consider a non-increasing function \( f \) on interval \([a, b]\), such as the one depicted in Figure 3. Consider as well an arbitrary point \( x_0 \in [a, b] \) and an arbitrary \( \delta > 0 \).

The value of \( F \) at \( x_0 \) is equal to the gray area in Figure 3. The black area in Figure 3 is the incremental area corresponding to \( x_0 + \delta \), and the blue area is the further incremental area corresponding to \( x_0 + 2\delta \).

Since \( f \) is positive, the value of \( F \) at \( x_0 + \delta \) (both the gray and the black areas), must exceed the value of \( F \) at \( x_0 \) (only the gray area); likewise for the value of \( F \) at \( x_0 + \delta \) relative to the value of \( F \) at \( x_0 + 2\delta \). So \( F \) is increasing. Moreover, since \( f \) is non-increasing, the blue area is no larger than the black area. These observations ensure that every line segment joining arbitrary points on \( F \) lies entirely below \( F \). So \( F \) is concave. (See Figure 4.)

![Figure 3: Decreasing function \( f(x) \), where \( x \in [0, 1] \).](image)

In sum, the integral of the virtual value function (in quantile space), namely the revenue curve, is necessarily concave.

## 5 Posted-Price Mechanisms

We now return to our regularly scheduled program: Our goal is to derive an approximation ratio for the simple mechanism, “post a random price,” in the single bidder setting, assuming \( F \) is regular.

We will analyze this mechanism not by drawing a random price from \( F \), but equivalently, by drawing a random quantile from \( U(0,1) \),
and posting price \( v(q) \). The expected revenue of this mechanism, \( \text{APX} \), is the area under the revenue curve in quantile space.

Let \( q^* \) be the quantile corresponding to the optimal posted price, \( \pi^* \): i.e., \( v(q^*) = \pi^* \). The expected revenue in quantile space generated by posting price \( \pi^* \) is \( \text{OPT} = R(q^*) \). We can depict this quantity by drawing a box of height \( R(q^*) \) and width 1, as shown in Figure 5.

\( \text{OPT} \) upper bounds \( \text{APX} \). To lower bound \( \text{APX} \), observe that the area under the revenue curve is at least the area of the triangle with vertices \((0, 1), (1, 0), \) and \( (q, R(q^*)) \) (see Figure 5). The area of this triangle is half the area of the box, and hence half the value of \( \text{OPT} \).

Therefore, posting a price that is simply a random draw from \( F \), yields, in expectation, at least half the total expected revenue of the optimal posted-price mechanism: i.e., \( \text{APX} \geq \frac{1}{2} \text{OPT} \).
5.1 Infinite Supply

Assume an infinite supply of copies of some good: e.g., a digital good, such as an audio or video recording.

Assume further that there are multiple potential buyers for this good, each of whom draws its value from the distribution $F$ (i.e., values are i.i.d. draws from $F$). What is the total expected revenue of posting a price randomly drawn from $F$?

Using our earlier analysis, we can expect to generate at least half the optimal revenue from each individual bidder, and since values are i.i.d., we conclude that this mechanism, in the infinite-supply setting, yields an approximation ratio of $1/2$.

5.2 Prior-Independent Mechanisms

We end this lecture a simple modification that yields a prior-independent mechanism: i.e., one that is independent of $F$.

The modified mechanism is as follows:

1. Collect sealed bids from each bidder.
2. Select a bidder $j$ uniformly at random.
3. Remove bidder $j$ from the mechanism.
4. Set a reserve price to $v_j$ for each bidder $N \setminus j$.
5. Allocate to every bidder that meets reserve, and charge them $v_j$.

How well does this mechanism do? Let $\text{APX}$ denote the total expected revenue of this mechanism, and let $\text{OPT}$ denote the total expected revenue of the optimal mechanism.

The way this mechanism selects a reserve price $v_j$ is equivalent to drawing a random value from the distribution $F$. Indeed, from the point of view of each bidder in $N \setminus j$, it remains the case that the reserve price is some randomly sampled value from $F$. But now only $n - 1$ bidders can pay, so the approximation ratio is:

$$\frac{\text{APX}}{\text{OPT}} \geq \frac{1}{2} \left( \frac{n - 1}{n} \right).$$

We can improve the approximation ratio by changing the way we set reserve prices. For example, we can offer bidder $j$ a reserve price equal to the value some other bidder $i \neq j$ submits. Again, from the point of view of bidder $j$, this reserve price is some randomly sampled value from $F$. With this modification, bidder $j$’s contribution to total expected revenue is the same as all the other bidders, and we recover the original approximation ratio of $1/2$. 