Revenue-Maximizing Auctions
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We present a mathematical program and a pointwise approach to solving for optimal (i.e., revenue-maximizing) single-good auctions.

1 Mathematical Program

Recall that revenue maximization in a single-good auction can be described as a mathematical program with the following constraints:

1. incentive compatibility
2. individual rationality
3. allocation constraints
4. ex-post feasibility

This program can be stated formally as follows:

$$\max_{x, p} \mathbb{E}_{v \sim F} \left[ \sum_{i \in N} p(v_i, v_{-i}) \right]$$

subject to

$$v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i x_i(t_i, v_{-i}) - p_i(t_i, v_{-i}) \quad \forall i \in N, \forall v \in T$$

$$v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq 0 \quad \forall i \in N, \forall v \in T$$

$$0 \leq x_i(v_i, v_{-i}) \leq 1 \quad \forall i \in N, \forall v \in T$$

$$\sum_{i \in N} x_i(v_i, v_{-i}) \leq 1 \quad \forall v \in T$$

2 Making Use of Myerson’s Results

Myerson’s lemma tells us that we can express the IC and IR constraints with monotonicity of the allocation rule and a particular payment rule instead:

$$x_i(v_i, v_{-i}) \geq x_i(t_i, v_{-i}), \quad \forall i \in N, \forall v_i \geq t_i \in T_i$$

$$p_i(v_i, v_{-i}) = v_i x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(z, v_{-i}) \, dz, \quad \forall i \in N, \forall v \in T.$$ 

Myerson’s theorem tells us that we can replace the objective of maximizing total expected payments with the equivalent objective of maximizing total expected virtual welfare.

Analogous to the case of welfare, Myerson’s results suggest a two-step process to solving for a revenue-maximizing auction:
1. First, design an algorithm that can find an ex-post feasible, virtual-welfare-maximizing allocation given $v \in T$.

2. If it so happens that monotonicity holds (i.e., the allocation algorithm is monotonic in values), construct truthful payments using Myerson’s payment formula.

   Monotonicity of the allocation function holds precisely when the virtual value function is non-decreasing in values. In turn, this property of the virtual value function holds when the underlying distribution in terms of which it is defined is so-called regular.

   

Algorithm 1: Revenue Maximization.
This algorithm proceeds pointwise, meaning one value vector at a time.

1: for all $v \in T$ do
  \quad \triangleright \text{Find allocations}
2: \quad for all $i \in N$ do
3: \quad \quad $x_i(v_i, v_{-i}) \leftarrow 0$
4: \quad end for
5: \quad if $\max_i \{\psi_i(v_i) : i \in N\} \geq 0$ then
6: \quad \quad $w(v) \leftarrow \arg \max_i \{\psi_i(v_i) : i \in N\}$
7: \quad \quad for all $i^* \in w(v)$ do
8: \quad \quad \quad $x_{i^*}(v_{i^*}, v_{-i^*}) \leftarrow 1/|w(v)|$
9: \quad \quad end for
10: \quad end if
11: end for
12: for all $i \in N$ do
  \quad \triangleright \text{Compute payments}
13: \quad for all $v \in T$ do
14: \quad \quad $p_i(v_i, v_{-i}) \leftarrow v_i x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(z, v_{-i}) \, dz$
15: \quad end for
16: end for
17: $S \leftarrow \sum_{i=1}^n \mathbb{E}_{v \sim F}[x_i(v_i, v_{-i})]$ \hspace{1cm} \triangleright \text{Total expected welfare}
18: $R \leftarrow \sum_{i=1}^n \mathbb{E}_{v \sim F}[p_i(v_i, v_{-i})]$ \hspace{1cm} \triangleright \text{Total expected revenue}
19: return $S, R, x, p$