Regular and MHR Distributions
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We define the hazard rate function, and then regular and monotone hazard rate (MHR) distributions.

1 An Aside: Survival Distributions

Assume $T$ is a continuous random variable with CDF $F(t)$ and PDF $f(t) > 0$, indicating the probability an event (e.g., marriage, parenthood, migration, death, etc.) has occurred by time $t \geq 0$. So,

$$F(t) = \Pr(T \leq t) = \int_0^t f(x)dx.$$ 

We also define the survival distribution $S(t)$, which indicates the probability that the event has not occurred by time $t$:

$$S(t) = \Pr(T > t) = 1 - F(t) = 1 - \int_0^t f(x)dx = \int_t^\infty f(x)dx$$

The hazard rate\footnote{Also called a failure rate} function $h(t)$ describes the instantaneous rate of occurrence of the event:

$$h(t) = \lim_{\delta \to 0} \frac{\Pr[t < T \leq t + \delta | T > t]}{\delta}.$$

The numerator in this expression is the probability that the event will occur in the interval $(t, t + \delta]$, given that it has not occurred by time $t$, and the denominator is the width of the interval. Hence, this fraction describes the rate of occurrence of the event, per unit of time. Taking the limit as $\delta \to 0$ yields the instantaneous rate of occurrence.

The hazard rate function can be simplified as follows:

$$h(t) = \lim_{\delta \to 0} \frac{\Pr[t < T \leq t + \delta | T > t]}{\delta}$$

$$= \lim_{\delta \to 0} \left( \frac{\Pr[t < T \leq t + \delta, T > t]}{\delta} \right) \left( \frac{1}{\Pr[T > t]} \right)$$

$$= \lim_{\delta \to 0} \left( \frac{\Pr[t < T \leq t + \delta]}{\delta} \right) \left( \frac{1}{\Pr[T > t]} \right)$$

$$= \frac{f(t)}{S(t)}$$

$$= \frac{f(t)}{1 - F(t)}$$

This latter form often rears its\footnote{no-longer-seeming-so-ugly} head in auction analyses.
Usually, a hazard rate is assumed to be increasing, decreasing, or constant. An increasing hazard rate signifies that the unit is becoming more and more prone to failure, with time. A decreasing hazard rate means the opposite: the unit is improving with time. Other possibilities include a U-shaped, or an upside-down U-shaped, hazard rate. The former is often used to model a human life span, because early in life we are very vulnerable, while at mid-life risks level off, until later in life when we are vulnerable again.

2 Regular and MHR Distributions

Observe that the virtual value function of a distribution $F$ relates to the corresponding hazard rate function $h$ as follows:

$$\varphi(v) = v - \frac{1 - F(v)}{f(v)} = v - \frac{1}{h(v)}$$

Now recall that Myerson's optimal auction design recipe requires that the virtual value function be non-decreasing in values: i.e., for $v \geq t \in T$, $\varphi(v) \geq \varphi(t)$. Distributions for which the corresponding virtual value function is non-decreasing are called regular.

A related, and stronger, condition is that the hazard rate function be non-decreasing: i.e., for $v \geq t \in T$, $h(v) \geq h(t)$. This condition is called the monotone hazard rate (MHR) condition. MHR implies regularity, but the two are not equivalent.

**Proposition 2.1.** MHR implies regularity.

**Proof.** If $h(v)$ is non-decreasing, then

$$\frac{1}{h(v)} = \frac{1 - F(v)}{f(v)}$$ (1)

is non-increasing. So, for $v \geq t$,

$$\frac{1 - F(v)}{f(v)} \leq \frac{1 - F(t)}{f(t)}.$$ (2)

Likewise,

$$-\frac{1 - F(v)}{f(v)} \geq -\frac{1 - F(t)}{f(t)}.$$ (3)

It follows that the virtual value function is non-decreasing:

$$v - \frac{1 - F(v)}{f(v)} \geq v - \frac{1 - F(t)}{f(t)} \geq t - \frac{1 - F(t)}{f(t)},$$ (4)

Therefore, MHR implies regularity. □
To show that the sets of MHR and regular distributions are distinct, we present an example of a distribution that satisfies regularity, but not MHR. This distribution is said to have “heavy tails.”

**Example 2.2 (Regular, and not MHR).** The distribution

\[ F(v) = 1 - \frac{1}{v + 1}, \quad (5) \]

has density

\[ f(v) = \frac{1}{(v + 1)^2}, \quad (6) \]

and support \((0, \infty)\). The CDF and PDF are shown in Figure 1.

The hazard rate function is

\[ h(v) = \frac{f(v)}{1 - F(v)} \]

\[ = \frac{1}{(v+1)^2} \]

\[ = \frac{1}{1 - \left(1 - \frac{1}{v+1}\right)} \]

\[ = \frac{1}{v + 1}. \quad (9) \]

Since \( h(v) \) is a strictly decreasing function, \( F \) does not satisfy the MHR condition. However, the virtual value function is non-decreasing:

\[ \varphi(v) = v - \frac{1}{h(v)} \]

\[ = v - (v + 1) \]

\[ = -1. \quad (12) \]

Therefore, \( F \) satisfies the regularity, but not the MHR, condition.

Although \( F \) is regular, Myerson’s scheme for allocating only to bidders with non-negative virtual values would not maximize revenue in an auction where values are distributed according to \( F \), as no good would ever be allocated, and revenue would always be 0. The issue is that \( F \)’s support is infinite.