Order Statistics and Revenue Equivalence

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We derive the first- and second-order statistics for the uniform distribution on [0, 1]. We use these results to prove that the expected revenue of the first-price auction is equal to that of the second-price auction.

**Definition 0.1.** The kth-order statistic, denoted $X_{(k)}$, of a statistical sample is the kth-largest draw (among n).

The first-order statistic is the maximum of n draws, while the nth-order statistic is the minimum of n draws.

1 First-Order Statistic

We now compute the expected value of $X_{(1)}$, the first order statistic, when sampling i.i.d. a uniform distribution on [0, 1]. That is,

$$
\mathbb{E} \left[ X_{(1)} \right] = \int_{0}^{1} x f_{X_{(1)}}(x) \, dx.
$$

We start by observing that for some $x \in [0, 1]$, the CDF, $F_{X_{(1)}}(x) = \Pr(X_{(1)} \leq x) = x^n$, which is simply the probability that all n draws are less than x.

By the definition of a CDF,

$$
F_X(x) = \int_{-\infty}^{x} f_X(t) \, dt.
$$

By the Second Fundamental Theorem of Calculus,

$$
f_X(x) = \frac{d}{dx} F_X(x).
$$

In particular,

$$
f_{X_{(1)}}(x) = \frac{d}{dx} F_{X_{(1)}}(x)
= \frac{d}{dx} x^n
= nx^{n-1}.
$$

Therefore,

$$
\mathbb{E} \left[ X_{(1)} \right] = \int_{0}^{1} x f_{X_{(1)}}(x) \, dx = \int_{0}^{1} nx^n \, dx = \frac{n}{n+1}.
$$
1.1 Second-Order Statistic

We follow the same steps computing the second-order statistic (using fewer words).

CDF:

\[ \Pr(X_{(2)} \leq x) = x^n + nx^{n-1}(1-x) \]

In words, all the samples can be less than \( x \), which happens with probability \( x^n \), or only \( n-1 \) of the samples can be less than \( x \) which can happen in \( n \) different ways, each with probability \( x^{n-1}(1-x) \).

PDF:

\[ f_{X_{(2)}}(x) = nx^{n-1} + n(n-1)x^{n-2}(1-x) - nx^{n-1} = n(n-1)x^{n-2}(1-x) \]

Expected value of the second-order statistic:

\[ E[X_{(2)}] = \int_0^1 xf_{X_{(2)}}(x)dx \]

\[ = n(n-1) \int_0^1 (x^{n-1} - x^n) dx \]

\[ = n(n-1) \left( \frac{1}{n} - \frac{1}{n+1} \right) \]

\[ = n(n-1) \frac{1}{n(n+1)} \]

\[ = \frac{n-1}{n+1} \]

2 Revenue Equivalence

**Theorem 2.1.** If bidder’s values are uniform, i.i.d., the expected revenue of the first-price auction is equal to that of the second-price auction, assuming bidders behave according to their respective equilibrium strategies.

**Proof.** The support of the uniform distribution does not matter; we choose \([0, 1] \) for convenience. Let \( R_1 \) and \( R_2 \) denote the expected revenue of the first- and second-price auctions, respectively.

In a first-price auction, the expected revenue is equal to the expected highest bid. Suppose the winning bidder is \( i^* \). Then since the equilibrium bid function \( b_i = \left( \frac{n-1}{n} \right) v_i \), it follows that

\[ R_1 = E[b_{i^*}] \]

\[ = E\left[ \left( \frac{n-1}{n} \right) v_{i^*} \right] \]

\[ = \left( \frac{n-1}{n} \right) E[v_{i^*}] \]
But $E[v_i]$ is the expected value of the maximum of $n$ i.i.d. random
variables uniformly distributed on $[0, 1]$, which is $\frac{n}{n+1}$. Thus,

$$R_1 = \left(\frac{n-1}{n}\right) \left(\frac{n}{n+1}\right) = \frac{n-1}{n+1}.$$

In the second-price auction, the bidder with the highest value
wins, paying the second-highest value. Therefore, the expected rev-
enue is equal to the expected value of the second-order statistic: i.e.,

$$R_2 = \frac{n-1}{n+1}.$$

Therefore, $R_1 = R_2$. ☑️

### A $k$th-Order Statistic

**Beta Function** The Beta function $B(x, y)$ is defined by the following
integral:

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dy.$$  

When $x$ and $y$ are positive integers, Beta simplifies as follows:

$$B(x, y) = \frac{(x-1)!(y-1)!}{(x+y-1)!}.$$

We will use the Beta function in (the very last step of) our derivation
of the $k$th-order statistic.

We begin by computing the probability the $k$th-order statistic lies
in some small interval $[x, x + \Delta x] \subset [0, 1]$. When $X$ is i.i.d.,

$$P(X_k \in [x, x + \Delta x]) = n \binom{n-1}{k-1} \cdot P(X \in [x, x + \Delta x]) \cdot P(X > x + \Delta x)^{k-1} + O(\Delta x^2)$$

The middle three probabilities are, respectively, the chance of:

- exactly $n - k$ values less than $x$,
- exactly one value between $x$ and $x + \Delta x$,
- and exactly $k - 1$ values greater than $x + \Delta x$.

This gives the probability of one specific arrangement of this form,
so we multiply by the number of possible arrangements. There are
$n$ possible agents who could have a value between $x$ and $x + \Delta$,
after which there are $\binom{n-1}{k-1}$ possible groups of agents who could
have values greater than $x$, after which the remaining $n - k$ agents
are fixed. **N.B.** There is also a chance that there are multiple values
between $x$ and $x + \Delta x$, but each such probability will contain a $\Delta x^i$
term, with $i \geq 2$. We capture these terms by adding in $O(\Delta x^2)$. 
The assumption of a uniform distribution on $[0, 1]$ yields the following further simplification:

$$P(X_{(k)} \in [x, x + \Delta x]) = n \binom{n-1}{k-1} x^{n-k} \cdot \Delta x \cdot (1 - x - \Delta x)^{k-1} + O(\Delta x^2)$$

Recall that discrete expectation is given by

$$\sum_{i=1}^{m} x_{i} P(X_{(k)} \in [x_{i}, x_{i+1}])$$

where $x_{i+1} = x_{i} + \Delta x$ (assume even-spaced intervals of length $\Delta x$). To calculate the corresponding continuous expectation, we take the limit as $m \to \infty$. When we do this, we throw away the $\Delta x$ terms, because they become arbitrarily small. Finally,

$$X_{(k)} = \lim_{m \to \infty} \sum_{i=1}^{m} x_{i} P(X_{(k)} \in [x_{i}, x_{i+1}])$$

$$= n \binom{n-1}{k-1} \left( \lim_{m \to \infty} \sum_{i=1}^{m} x_{i}^{n-k+1} \Delta x (1 - x_{i} - \Delta x)^{k-1} + O(\Delta x^2) \right)$$

$$= n \binom{n-1}{k-1} \int_{0}^{1} x^{n-k+1} (1 - x)^{k-1} dx$$

$$= n \binom{n-1}{k-1} B(n - k + 2, k)$$

$$= \frac{n!}{(k-1)!(n-k)!} \left( \frac{(n - k + 1)!(k-1)!}{(n+1)!} \right)$$

$$= \frac{n-k+1}{n+1}.$$