Auction Design Goals

Professor Greenwald
2018-02-07

We formally define the sealed-bid auction for a single good. This auction format is defined by two rules, an allocation rule and a payment rule. Using this formal framework, we state auction design goals.

1 Auction Model

Assume a set of \( n \geq 2 \) agents participating in an auction for one good. We adopt the IPV model, so that each agent \( i \in N = \{1, \ldots, n\} \) has value \( v_i \) drawn from continuous distribution \( F_i \), with support \( T_i = [v_i, \overline{v}_i] \) for some \( v_i, \overline{v}_i \in \mathbb{R}_+ \), which describes how much they value the good.\footnote{For simplicity, we sometimes assume symmetry, so that \( F_i = F, \overline{v}_i = \overline{v}, \) and \( v_i = 0, \) for all bidders \( i \in N.\)} We call this environment single-parameter, because bidders are characterized by but one parameter—their values.

Each agent \( i \) employs a strategy \( s_i : T_i \rightarrow B_i \), which, as usual, is a function from their space of possible values to their space of possible actions, which, in an auction, are bids. It is assumed that bids are submitted to the auctioneer in sealed envelopes, so that no bidder knows what any other has bid. Given a profile of values \( \mathbf{v} = (v_1, \ldots, v_n) \), the vector of bids submitted to the auctioneer is denoted \( \mathbf{b} = (b_1, \ldots, b_n) \equiv (s_1(v_1), \ldots, s_n(v_n)) \).

Given a bid profile, the auctioneer computes allocations and payments according to the auctions’ rules. The allocation rule describes how the winner(s) of the auction is determined; the payment rule describes what the participants pay.

Notationally, the probability that each agent wins is written \( x(\mathbf{b}) = (x_1(\mathbf{b}), \ldots, x_n(\mathbf{b})) \), and the payment that each agent makes is written \( p(\mathbf{b}) = (p_1(\mathbf{b}), \ldots, p_n(\mathbf{b})) \). In particular, the probability that \( i \) wins the auction is \( x_i(\mathbf{b}) \in [0, 1] \), and the payment that \( i \) makes to the auctioneer is \( p_i(\mathbf{b}) \in \mathbb{R}_+ \). We call the pair \( (x(\mathbf{b}), p(\mathbf{b})) \) the outcome of the auction.

Finally, we assume a quasi-linear model of agent utility,\footnote{A quasi-linear function is linear in one variable, called the numeraire. A numeraire is a basic standard for measuring value (e.g., money).} where \( u_i(\mathbf{b}; v_i) = v_i x_i(\mathbf{b}) - p_i(\mathbf{b}) \). We almost always abbreviate utility as follows: \( u_i(\mathbf{b}) \equiv u_i(\mathbf{b}; v_i) \). Further, when the bidding profile is clear from context, we often write: \( u_i = v_i x_i - p_i \).

First- and Second-Price Auctions In the two sealed-bid auctions of primary interest, namely first- and second-price auctions, the space of possible bids contains the space of possible values: i.e., \( T_i \subseteq B_i \), for all bidders \( i \in N \). Upon receiving bids in sealed envelopes, the auctioneer selects as the winner a bidder with the high-
Est bid, \( i^* \in \arg \max_{i \in N} b_i \), allocating to this winner with probability \( x_i^* (b_i, b_{-i^*}) = 1 \), and breaking ties, as necessary.\(^3\)

Let \( b_{(k)} \) denotes the \( k \)-th order statistic, meaning the \( k \)-th largest draw among \( n \) samples. In particular, \( b_{(n)} \) and \( b_{(n-1)} \) denote the highest and second-highest bids, respectively.

In the first-price auction, the winner of the auction, \( i^* \), is charged their bid (i.e., the highest bid), \( p_i (b_{i^*}, b_{-i^*}) = b_{(n)} \), and all other bidders \( i \neq i^* \in N \) are charged \( p_i (b_i, b_{-i}) = 0 \).

In the second-price auction, the winner of the auction, \( i^* \), is charged the second-highest bid, \( p_i (b_{i^*}, b_{-i^*}) = b_{(n-1)} \), and all other bidders \( i \neq i^* \in N \) are charged \( p_i (b_i, b_{-i}) = 0 \).

\(^2\) Design Goals

There are (at least) three desirable properties of an auction:

**Incentive Guarantees** The first goal has to do with incentives. In order to compare competing auction designs, we must be able to predict the outcome of an auction, which in turn means, predicting the strategic behavior of the agents in the auction. If we make our auctions simple for agents to reason about, then we may have a better chance of predicting what agents will do. Bidding truthfully is always an option, so one way to make things simple is to ensure that bidding truthfully is an equilibrium strategy. When this property holds, an auction is said to be **incentive compatible**.

There are two primary forms of incentive compatibility, one based on the notion of DSE, and a second, based on the notion of Bayes-Nash equilibrium. To arrive at the latter, we simply plug the truthful bidding strategy profile into the Bayes-Nash equilibrium definition.

**Definition 2.1.** If \( v \) denotes the bidders’ true values, then the **truthful bidding** strategy is as follows: \( s_i(v_i) = v_i \), for all bidders \( i \in N \).

**Definition 2.2.** An auction is called **Bayesian incentive compatible** (BIC) if truthful bidding is a Bayes-Nash equilibrium: i.e., for all bidders \( i \in N \) and for all (true) values \( v_i \in T_i \),

\[
\mathbb{E}_{v_{-i} \sim F_{-i}} [u_i(v_i, v_{-i})] \geq \mathbb{E}_{v_{-i} \sim F_{-i}} [u_i(s'_i(v_i), v_{-i})], \quad \forall s'_i \in S_i.
\]

In words, an auction is BIC if bidding truthfully is an interim (or equivalently, ex-ante) best response for each bidder, assuming all the other agents bid truthfully.

Similarly, we can plug the truthful bidding strategy profile into the ex-post Nash equilibrium (EPNE) definition.
Definition 2.3. An auction is called ex-post (Nash) incentive compatible (EPIC) if truthful bidding is a EPNE: i.e., for all bidders \(i \in N\) and for all (true) value profiles \(v \in T\),
\[
u_i(v_i, v_{-i}) \geq u_i(s'_i(v_i), v_{-i}), \quad \forall s'_i \in S_i.
\]
In words, an auction is EPIC if bidding truthfully is an ex-post best response for each bidder, assuming all the other agents bid truthfully.

A stronger form of incentive compatibility aligns with the notion of dominant strategy equilibrium.

Definition 2.4. An auction is called dominant strategy incentive compatible (DSIC) if truthful bidding is a DSE: i.e., for all bidders \(i \in N\) and for all (true) value profiles \(v \in T\),
\[
u_i(v_i, b_{-i}) \geq u_i(s'_i(v_i), b_{-i}), \quad \forall s'_i \in S_i \forall b_{-i} \in B_{-i}.
\]
In words, an auction is DSIC if bidding truthfully is an ex-post best response for each bidder, regardless of how the other agents bid.

Remark 2.5. In a direct mechanism, EPIC is equivalent to DSIC.

Proof. DSIC implies EPIC, so it suffices to show that EPIC also implies DSIC. Let \(M\) be an EPIC, one-shot mechanism such that the space of possible actions equals the space of possible values.

Since \(M\) is EPIC, for all bidders \(i \in N\) and for all (true) value profiles \(v \in T\), the truthful bidding profile \(s \in S\) satisfies
\[
u_i(v_i, v_{-i}) \geq u_i(s'_i(v_i), v_{-i}), \quad \forall s'_i \in S \forall v_{-i} \in T_{-i}.
\]

Our goal is to show that \(M\) is also DSIC: i.e., for all bidders \(i \in N\) and for all (true) values \(v_i \in T_i\),
\[
u_i(v_i, b_{-i}) \geq u_i(s'_i(v_i), b_{-i}), \quad \forall s'_i \in S_i \forall b_{-i} \in B_{-i}.
\]

But since \(B_i = T_i\) by assumption (i.e., the mechanism is direct), it suffices to show: for all bidders \(i \in N\) and for all (true) values \(v_i \in T_i\),
\[
u_i(v_i, v_{-i}) \geq u_i(s'_i(v_i), v_{-i}), \quad \forall s'_i \in S_i \forall v_{-i} \in T_{-i}.
\]

Fix a bidder \(i\) and their true value \(v_i\). For two arbitrary value profiles \(v'_{-i}, v''_{-i} \in T_i\), since \(M\) is EPIC,
\[
u_i(v_i, v'_{-i}) \geq u_i(s'_i(v_i), v'_{-i}), \quad \forall s'_i \in S_i,
\]
\[
u_i(v_i, v''_{-i}) \geq u_i(s'_i(v_i), v''_{-i}), \quad \forall s'_i \in S_i.
\]
Hence, truthful bidding is a best response for bidder \(i\), assuming others are also bidding truthfully, but regardless their value profiles. In other words, truthful bidding is a best response for bidder \(i\), for all other-agent value profiles: i.e.,
\[
u_i(v_i, v_{-i}) \geq u_i(s'_i(v_i), v_{-i}), \quad \forall s'_i \in S_i \forall v_{-i} \in T_{-i}.
\]

Since bidder \(i\) was arbitrary, truthful bidding is a dominant-strategy equilibrium (i.e., it is a best response for all bidders). \(\square\)
Economic Performance Guarantees  The second goal is that the auction format achieve some objective. A popular choice is welfare maximization.\textsuperscript{4} Welfare is defined as the total expected utility of all participants, including the auctioneer, which assuming incentive compatibility, can be written as follows:

\[
E_v \left[ \sum_{i \in N} u_i(v) + \sum_{i \in N} p_i(v) \right] = E_v \left[ \sum_{i \in N} (v_i x_i(v) - p_i(v)) + \sum_{i \in N} p_i(v) \right] \\
= E_v \left[ \sum_{i \in N} v_i x_i(v) \right]
\]

Revenue maximization is a popular alternative (Think Google, etc.). Revenue is defined as the total expected payments, only:

\[
E_v \left[ \sum_{i \in N} p_i(v) \right]
\]

Computational Performance Guarantees  In addition to aiming for economic efficiency, we also aim to design auctions that are computationally efficient, meaning they run in polynomial time and space.

The allocation algorithm for the first- and second-price auctions—namely, allocate to the highest bidders—satisfies this requirement: it is $O(n)$ in time and space in the worst case. There may be an $n$-way tie, and a tie-breaking rule may randomly select among all the tied bidders. Hence, storing the set of all possible winners requires $O(n)$ space in the worst case.

Similarly, calculating payments in the second-price auction is $O(n)$, and requires $O(n)$ space in the worst case. In contrast, calculating payments in a first-price auction is $O(1)$. Hence, there are computational differences among different auction formats, rendering computation relevant to our auction analyses.

3  A Welfare-Maximizing Mathematical Program

Solving for the welfare-maximizing $k$-good auction can be viewed as a constrained optimization problem.

The function being maximized is total expected welfare:

\[
E_{v \sim F} \left[ \sum_{i \in N} v_i x_i(v_i, v_{-i}) \right]
\]

The decision variables are the allocation rule $x$ and payment rule $p$.

The constraints are as follows:\textsuperscript{5}

1. Incentive compatibility. Truthful bidding maximizes utility,\textsuperscript{6}

\[
u_i(v_i, v_{-i}) \geq u_i(t_i, v_{-i}), \quad \forall i \in N, \forall v_i, t_i \in T_i, \forall v_{-i} \in T_{-i};
\]

\textsuperscript{4} When welfare is maximized, an economy is said to be efficient.

\textsuperscript{5} Because of incentive compatibility, we assume everyone bids their true value: i.e., $b_i = v_i, \forall i \in N$.

\textsuperscript{6} Here, we are assuming $B_i = T_i, \forall i \in N$. 

2. **Individual rationality.** Utility is non-negative (assuming truthful bidding, which is ensured by the IC constraints):

\[ u_i(v_i, v_{-i}) \geq 0, \quad \forall i \in N, \forall v \in T; \]

3. **Allocation constraints.** The allocation variables vary with the auction set up. In a \( k \)-good auction, they are 0/1 variables in principle, but as there could be ties, we represent them as probabilities. The probability of winning must be in \([0, 1]\):

\[ 0 \leq x_i(v_i, v_{-i}) \leq 1, \quad \forall i \in N, \forall v \in T. \]

4. **Ex-post feasibility.** Goods are not overallocated:

\[ \sum_{i \in N} x_i(v_i, v_{-i}) \leq 1, \quad \forall v \in T; \]

Together this objective function and these constraints comprise a mathematical program that can be used to solve for an optimal \( k \)-good auction. The good news is, the objective function and the constraints are linear. The bad news is, there are an exponential number of constraints (assuming we discretize the value space). But do not despair. Roger Myerson won a Nobel prize in part for his elegant solution to this auction design problem. Stay tuned!