We introduce a new multi-parameter setting, for which we can find an approximately EPIC welfare-maximizing auction. We prove the EPIC property by making use of our recipe for doing so: 1. we prove sincere bidding in the auction yields an approximate VCG outcome, and 2. we show consistent bidding strategies dominate inconsistent ones.

1 A Multi-Unit Auction

We introduce a new scenario, a multi-unit auction for identical copies of a good, but rather than bidders having unit-demand valuations, here each bidder’s marginal value for an additional copy of the good is non-increasing. Precisely,

- We assume \( n \) bidders and \( m \) identical goods, with bidders indexed by \( i \), and goods, by \( j \).
- Each bidder \( i \) has a marginal value \( \mu_i(j) \) for their \( j \)th copy of the good.
- Each bidder \( i \)'s marginal values are non-increasing: \( \mu_i(1) \geq \mu_i(2) \geq \cdots \geq \mu_i(m) \).

Our goal is to construct an approximately EPIC welfare-maximizing ascending auction for this scenario. As a sanity check, we begin by constructing a direct DSIC welfare-maximizing auction.

1.1 The Direct Mechanism

Our present goal is to construct a direct DSIC welfare-maximizing auction that operates in polynomial time. Recall that if we cannot do so, then we cannot hope to construct a computationally efficient indirect EPIC welfare-maximizing auction for this scenario.

In the direct setting, the welfare-maximizing allocation can be computed via a simple greedy allocation algorithm:

- Collect a vector of bids \( b_i \) from all bidders \( i \in N \), with bid \( b_i(j) \) representing \( i \)'s bid on the \( j \)th copy of the good.
- Sort the bids, and then allocate the goods to the bidders who submitted the highest \( m \) bids, breaking ties arbitrarily.

For example, if \( b_i(4) \) is among the highest \( m \) bids, but \( b_i(5) \) is not, then bidder \( i \) is allocated four goods.
As usual, to achieve the VCG outcome, we combine this allocation algorithm with payments that charge bidders their externalities. For each bidder $i$, we sort the bids by bidders other than bidder $i$ from greatest to least, and then establish the following groupings.

\[
\begin{array}{cccccc}
\beta_1 & \beta_2 & \cdots & \beta_{m-x_i} & \beta_{m-x_i+1} & \beta_{m-x_i+2} & \cdots & \beta_m & \beta_{m+1} & \beta_{m+2} & \cdots & \beta_{mn}
\end{array}
\]

The bids in group $A$ are those that are allocated regardless of $i$'s presence. The bids in group $C$ are those that are not allocated regardless of $i$'s presence. The bids in group $B$ are those whose allocation depends on $i$'s presence. These bids comprise bidder $i$'s externality. We therefore charge bidder $i$, in total, for all $x_i$ goods in group $B$, the sum of these bids: i.e.,

\[
p_i(x_i) = \sum_{j=1}^{x_i} \beta_{m-x_i+j}.
\]

By charging each bidder their externality, we charge the VCG payments, thereby guaranteeing the DSIC property.

Observe that when bidder $i$ is allocated $x_i$ copies of the good, their VCG payment is the sum of $x_i$ bids, one per copy of the good. We can interpret these bids as follows: the smallest bid is bidder $i$’s payment for their first copy of the good; the second-smallest bid is their payment for their second copy of the good; and so on. Note that payments for additional copies are non-increasing, although values are (by assumption) non-decreasing. Payments are non-increasing because the first copy allocated to bidder $i$ displaces only bid $\beta_m$, whereas the second copy displaces bid $\beta_{m-1} \geq \beta_m$, and so on.

Building on these observations, we can express bidder $i$’s per-good VCG payments in terms of the other bidders’ demand sets. Define bidder $k$’s demand set at price $q$, $D_k(q) = \max \{ j \leq m \mid v_k(j) \geq q \}$.

Now, the price of bidder $i$’s $j$th copy is given by:

\[
p_i(j) = \inf \left\{ q \left| \sum_{k \neq i} D_k(q) \leq m - j \right. \right\}
\]

Note that these prices are non-decreasing! Each additionally copy of the good costs no less than the previous.

1.2 The Clinching Auction

Having satisfied the precondition for potential success, we now set our sights on an EPIC welfare-maximizing ascending auction. We present the following auction, called the clinching auction:\(^1\)

- Initialize $q = 0$.\(^1\)

• Collect demand sets from all bidders. (Initially, when \( q = 0 \), it
should be that \( D_i(q) = m \), for all bidders \( i \).)

• Alternate between incrementing \( q \) by \( \epsilon \) and collecting demand sets
from bidders until \( \sum_{i=1}^{n} D_i(q) \leq m \).

• Allocate to bidder \( i \) their final \( D_i(q) \) goods. If unallocated goods
remain, allocate them randomly to bidders \( i \) with leftover demand
at price \( q - \epsilon \): i.e., bidders \( i \) for whom \( D_i(q - \epsilon) - D_i(q) > 0 \).

• Charge bidder \( i \) (with \( \epsilon \) of) her externality. Specifically, charge
bidder \( i \) for her \( j \)th good:

\[
q_i(j) = -\epsilon + \min_{t \in \mathbb{Z}^+} \left\{ et \mid \sum_{k \neq i} D_k(et) \leq m - j \right\}.
\]

**Example 1.1.** Here is an example of a run-through of the clinching
auction lifted from the paper that introduced it. The example
is loosely based on the first US Nationwide Narrowband spectrum
auction in where there were five bidders and five licenses, with the
constraint that no bidder could win more than three licenses.

The bidders’ marginal values are as follows:

<table>
<thead>
<tr>
<th>License</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>123</td>
<td>74</td>
<td>125</td>
<td>84</td>
<td>44</td>
</tr>
<tr>
<td>Second</td>
<td>113</td>
<td>5</td>
<td>125</td>
<td>64</td>
<td>24</td>
</tr>
<tr>
<td>Third</td>
<td>103</td>
<td>3</td>
<td>49</td>
<td>7</td>
<td>5</td>
</tr>
</tbody>
</table>

The auction then proceeds as follows, with demands depicted only
at the most relevant prices:

<table>
<thead>
<tr>
<th>Price</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>25</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>45</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>50</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>65</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>75</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>85</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

At price 65, the demands of all bidders other than bidder A falls
below the total supply of 5. Hence, bidder A clinches their first li-
cense at this price. The license is said to be “clinched” because the
fact that the other bidders’ demands have fallen below 5 guarantees that bidder A will win this license.

At price 75, the demands of all bidders other than bidder A falls below 4, so A clinches their second license at this price. In addition, the demands of all bidders other than bidder C falls below 5, so C clinches their first license at this price. The auction terminates at price 85, when total demand meets total supply. At this price, bidder A clinches their third license, and bidder C, their second.

In sum, bidder A pays 65 + 75 + 85 for its three licenses, and bidder C pays 75 + 85 for its two licenses. Observe that prices on additional licenses are non-decreasing (in fact, in this example, they are strictly increasing).

The outcome of the clinching auction in this example (and always; see Proposition 1.2) is efficient (up to $m\epsilon$). In contrast, in this example, simultaneous ascending auctions would not have yielded an efficient outcome, as it would have been in bidder A’s best interest to decrease its demand to two licenses when the price was $75 rather than win all three licenses for $85 each. Winning only two licenses, A’s utility would have been $236 - 150 = 86$, whereas winning all three, A’s utility would have been $339 - 255 = 84$.

En route to proving that the clinching auction is EPIC, we will first argue that sincere bidding in this auction yields an outcome that is approximately VCG. Since its payments are approximately VCG by design, it suffices to show that it approximately welfare-maximizing.

**Proposition 1.2.** Assuming sincere bidding, the clinching auction generates within $m\epsilon$ of the maximum possible welfare.

**Proof.** Let $A$ denote the highest $m$ marginal values among all bidders,\footnote{breaking ties arbitrarily} and let $B$ be the marginal values corresponding to the final allocation. Note that regardless of who the precise winners are, payments do not change. Hence, it suffices to show that the difference between the total value of $A$ and $B$ is bounded above by $m\epsilon$.

Call the final round $k$. Assuming sincere bidding, the set of winning marginal values $S_k$ at round $k$ is a subset of $B$, because demand may fall below $m$ in the $k$th round, so that some marginal values in $S_{k-1}$ are allocated. (In the worst case, $S_k = \emptyset$.) Additionally,

- every marginal value in $i \in A \setminus S_k$ is at most $k\epsilon$—otherwise, it would be in $S_k$; and
- every marginal value in $i \in A \setminus S_k$ is also at least $(k-1)\epsilon$, because bidding is sincere, the auction terminates at round $k$, and the marginal values in $A$ are the highest $m$ marginal values.
Thus, the maximal difference between the total value of $A$ and $B$ is:

$$\sum_{i \in A \setminus B} v_i \leq \sum_{i \in A \setminus S_k} v_i \leq (|A| - |S_k|)(ke - (k - 1)e) = (|A| - |S_k|)\epsilon \leq m\epsilon.$$ 

The following theorem follows immediately:

**Theorem 1.3.** Sincere bidding in the clinching auction yields an outcome that is approximately VCG.

**Proposition 1.4.** The clinching auction is EPIC, up to $m\epsilon$.

**Proof.** We have already shown that the outcome of the clinching auction under sincere bidding is approximately VCG. Recalling our EPIC-proving strategy, we need only prove that inconsistent bidding cannot improve over consistent bidding.

Assume all bidders except bidder $i$ bid sincerely. Consequently, other bidders’ behaviors are not impacted by $i$’s strategy.

Moreover, $i$’s allocation and payments are dictated entirely by the other bidders’ demands, which, again $i$ cannot influence. So $i$ cannot impact what $i$ wins, nor the prices $i$ pays.

Under these circumstances, we argue that $i$ cannot benefit from bidding inconsistently. Bidding inconsistently in the clinching auction amounts to reporting false demand sets. All such reports are in fact consistent with some valuation or another. So bidding inconsistently in the clinching auction actually amounts to bidding consistently in each round with respect to varying underlying valuations.

The argument as to why bidding according to any valuation other than its own cannot benefit $i$ is the usual one.

- If $i$ bids as if their value for a copy of the good were less than it actually is (i.e., it does not demand the copy, even though the price is below its marginal value), and if they don’t win the copy as a result (someone else clinches it instead), they are unhappy.

- If $i$ bids as if their value for a copy of the good were more than it actually is (i.e., it reports demand for that copy at a price above its marginal value), and if they win (clinch) the copy as a result (at its high price), they are again unhappy.

In sum, inconsistent bidding, meaning bidding in some round as if their valuation were different than it actually is, cannot improve $i$’s utility over consistent bidding. 

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References