We present an example of a Bayesian game. This set of notes is partially based on this video.

1 Introduction

Two roommates, Alice and Bob, are planning to see a movie tonight, at one of two possible locations: the cinema (C), or at home (H). Alice is interested in Bob, and would like to be in the same place as Bob. However, we do not know if Bob is interested (I), or uninterested (U) in Alice.\footnote{And they don’t communicate beforehand, because reasons.} If Bob is also interested in Alice, then he also receives positive payoff for being with Alice. Conversely, if Bob is not interested in Alice, then he receives zero payoff for begin with Alice. Each of Bob’s types are equally likely (so Pr(I) = Pr(U) = $\frac{1}{2}$).

If Bob is interested in Alice, the utility Alice and Bob receive are given by Figure 1, where Alice is the row player, and Bob is the column player.

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>H</th>
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</thead>
<tbody>
<tr>
<td>C</td>
<td>10, 5</td>
<td>0, 0</td>
</tr>
<tr>
<td>H</td>
<td>0, 0</td>
<td>5, 10</td>
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Pr(I) = $\frac{1}{2}$

On the other and, if Bob is not interested in Alice, the utility Alice and Bob receive are given by Figure 2, where Alice is again the row player, and Bob, the column player.

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<tbody>
<tr>
<td>C</td>
<td>10, 0</td>
<td>0, 10</td>
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<tr>
<td>H</td>
<td>0, 5</td>
<td>5, 0</td>
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Pr(U) = $\frac{1}{2}$

2 Reducing this Bayesian Game to a Normal-form Game

The ex-ante expected utility of player $i$ assuming strategy profile $s$ is:

$$E[u_i(s)] = E_{t \sim F}[u_i(s, t)].$$

(1)

Using this formula, we can describe ex-ante expected utilities for any strategy profile in our Bayesian game.
For example, suppose Alice plays C, and Bob plays C if he has type I, and H if he has type U. (We use the notation CH as shorthand to describe Bob’s strategy.) Then, Alice’s expected utility is:

\[
\mathbb{E}[u_a(C, CH)] = \sum_{t \in T} \Pr(t) u_a(C(t_a), CH(t_b), t)
\]

\[
= \Pr(I) u_a(C(\cdot), CH(I), (\cdot, I)) + \Pr(U) u_a(C(\cdot), CH(U), (\cdot, U))
\]

\[
= \Pr(I) u_a(C, C, (\cdot, I)) + \Pr(U) u_a(C, H, (\cdot, U))
\]

\[
= \frac{1}{2} (10) + \frac{1}{2} (0)
\]

\[
= 5.
\]

And, Bob’s expected utility would be

\[
\mathbb{E}[u_b(CH, C)] = \sum_{t \in T} \Pr(t) u_b(CH(t_b), C(t_a), t)
\]

\[
= \Pr(I) u_b(CH(I), C(\cdot), (\cdot, I)) + \Pr(U) u_b(CH(U), C(\cdot), (\cdot, U))
\]

\[
= \Pr(I) u_b(C, C, (\cdot, I)) + \Pr(U) u_b(H, C, (\cdot, U))
\]

\[
= \frac{1}{2} (5) + \frac{1}{2} (10)
\]

\[
= \frac{15}{2}.
\]

We can continue in this fashion and describe the ex-ante expected utilities of Alice (Figure 3) and Bob (Figure 4).

<table>
<thead>
<tr>
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<th>CC</th>
<th>CH</th>
<th>HC</th>
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<tbody>
<tr>
<td>C</td>
<td>10</td>
<td>5</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>H</td>
<td>0</td>
<td>5/2</td>
<td>5/2</td>
<td>5</td>
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<tr>
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<th>CC</th>
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<tbody>
<tr>
<td>C</td>
<td>5/2</td>
<td>15/2</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>H</td>
<td>5/2</td>
<td>0</td>
<td>15/2</td>
<td>5</td>
</tr>
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This game has one pure-strategy Nash equilibria: C, CH, which leads to payoff 5 for Alice and 15/2 for Bob.\(^2\)

3 Finding Mixed Strategies

There, are potentially more equilibria in this game, namely mixed-strategy Nash equilibria. As neither player has any (necessarily pure) dominant strategies, they may employ mixed strategies to maximize their expected utility.

- Let \( p \) be the probability that Alice plays C.
Bayesian Battle of the Sexes

Let \( q_I \) be the probability that Bob plays C, if Bob is interested in Alice.

Let \( q_U \) be the probability that Bob plays C, if Bob is uninterested in Alice.

If Alice uses the mixed strategy \( p \), how should Bob respond? Well, this depends on whether Bob has type I or type U. We will start by assuming Bob has type I.

3.1 Starting Point: If Bob is Interested

If Bob is interested in Alice, his payoff for playing C is:

\[ 5p + 0(1 - p) = 5p. \]  \hfill (12)

His payoff for playing H is:

\[ 0p + 10(1 - p) = 10 - 10p. \]  \hfill (13)

What value of \( p \) for Alice makes Bob indifferent between his two actions? Equating the payoffs, and solving for \( p \) yields:

\[ 5p = 10 - 10p \]  \hfill (14)

\[ 15p = 10 \]  \hfill (15)

\[ p = \frac{2}{3} \]  \hfill (16)

Therefore, the mixed strategy \((2/3, 1/3)\) for Alice makes Bob indifferent between his two actions, if Bob has type I.

If Bob is uninterested in Alice, his payoff for playing C is:

\[ 0p + 5(1 - p) = 5(1 - p) \]  \hfill (17)

His payoff for playing H is:

\[ 10p + 0(1 - p) = 10p. \]  \hfill (18)

Plugging in Alice’s strategy yields a payoff of \( 5 - 5 \left( \frac{2}{3} \right) = \frac{5}{3} \) for playing C, and \( 10 \left( \frac{2}{3} \right) = \frac{20}{3} \) for playing H. Bob’s payoff if he has type U is strictly greater when playing H, so \( q_U = 0 \).

Alice’s payoff Alice for playing C, when \( q_U = 0 \), is:

\[
\Pr(I) \left[ 10q_I + 0(1 - q_I) \right] + \Pr(U) \left[ 10q_U + 0(1 - q_U) \right] \\
= \frac{1}{2} \left[ 10q_I \right] + \frac{1}{2} \left[ 10q_U \right] \\
= \frac{1}{2} \left[ 10q_I \right] + \frac{1}{2} \left[ 10(0) \right] \\
= 5q_I.
\]  \hfill (19)
Her payoff for playing H, when \( q_U = 0 \), is:

\[
Pr(I) [0q_I + 5(1 - q_I)] + Pr(U) [0q_U + 5(1 - q_U)]
\]

(23)

\[
= \frac{1}{2} [5(1 - q_I)] + \frac{1}{2} [5(1 - q_U)]
\]

(24)

\[
= \frac{1}{2} [5(1 - q_I)] + \frac{1}{2} [5(1)]
\]

(25)

\[
= \frac{5}{2} (1 - q_I) + \frac{5}{2}.
\]

(26)

What value of \( q_I \) for Bob makes Alice indifferent between her two actions? Equating the payoffs, and solving for \( q_I \) yields:

\[
5q_I = \frac{5}{2} (1 - q_I) + \frac{5}{2}
\]

(27)

\[
10q_I = 5(1 - q_I) + 5
\]

(28)

\[
15q_I = 10
\]

(29)

\[
q_I = \frac{2}{3}
\]

(30)

Putting it all together:

1. Alice plays C with probability \( p = \frac{2}{3} \), and H with probability \( 1 - p = \frac{1}{3} \).
2. If Bob has type I, then he plays C with probability \( q_I = \frac{2}{3} \), and H with probability \( 1 - q_I = \frac{1}{3} \).
3. If Bob has type U, then he plays C with probability \( q_U = 0 \), and H with probability \( 1 - q_U = 1 \).

We can describe this mixed strategy compactly, as follows:

\[
\begin{pmatrix}
\begin{pmatrix}
\frac{2}{3} & 1 \\
\text{Alice}
\end{pmatrix},
\begin{pmatrix}
\frac{2}{3} & 1 \\
\text{Type I}
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
\text{Type U}
\end{pmatrix}
\end{pmatrix}
\]

(31)

Verifying We now verify that this mixed strategy is in fact a Bayes-Nash equilibrium. Fixing Alice’s (Bob’s) strategy, it should be the case that Bob (Alice) cannot employ an alternative mixed strategy that yields strictly more utility.

Alice The expected utility Alice receives for playing C is:

\[
u_a(C) = Pr(I) [10 (q_I) + 0 (1 - q_I)] + Pr(U) [10 (q_U) + 0 (1 - q_U)]
\]

(32)
The expected utility Alice receives for playing $H$ is:

$$u_a(H) = \Pr(I) \left[ 0 \left( q_I \right) + 5 \left( 1 - q_I \right) \right] + \Pr(U) \left[ 0 \left( q_U \right) + 5 \left( 1 - q_U \right) \right]$$

$$= \frac{1}{2} \left[ 0 \left( \frac{2}{3} \right) + 5 \left( \frac{1}{3} \right) \right] + \frac{1}{2} \left[ 0 \left( 0 \right) + 5 \left( 1 - 0 \right) \right]$$

$$= \frac{10}{3}. \quad (33)$$

Since the expected utilities are equal, Alice is indifferent between playing $C$ and $H$, and cannot improve her expected utility by mixing.

**Bob (type $I$)** The expected utility Bob (type $I$) receives for playing $C$ is

$$u_b(C) = 5 \left( p \right) + 0 \left( 1 - p \right)$$

$$= 5p$$

$$= \frac{10}{3}. \quad (39)$$

The expected utility Bob (type $I$) receives for playing $H$ is

$$u_b(H) = 0 \left( p \right) + 10 \left( 1 - p \right)$$

$$= 10 \left( 1 - p \right)$$

$$= \frac{10}{3}. \quad (43)$$

Since the expected utilities are equal, Bob is indifferent between playing $C$ and $H$, and cannot improve his expected utility by mixing.

**Bob (type $U$)** The expected utility Bob (type $U$) receives for playing $C$ is

$$u_b(C) = 0 \left( p \right) + 5 \left( 1 - p \right)$$

$$= 5 \left( 1 - p \right)$$

$$= \frac{5}{3}. \quad (46)$$

The expected utility Bob (type $U$) receives for playing $H$ is

$$u_b(H) = 10 \left( p \right) + 0 \left( 1 - p \right)$$

$$= 10 \left( p \right)$$

$$= \frac{20}{3}. \quad (49)$$

Since the expected utility of playing $H$ is strictly larger than the expected utility of playing $C$, Bob will play $H$. 
3.2 Starting Point: If Bob is Uninterested

If Bob is uninterested in Alice, his payoff for playing C is:

\[ 0p + 5(1-p) = 5(1-p). \]  
\[ (50) \]

His payoff for playing H is:

\[ 10p + 0(1-p) = 10p. \]  
\[ (51) \]

What value of \( p \) for Alice makes Bob indifferent between his two actions? Equating the payoffs, and solving for \( p \) yields:

\[ 5 - 5p = 10p \]  
\[ (52) \]
\[ 5 = 15p \]  
\[ (53) \]
\[ p = \frac{1}{3} \]  
\[ (54) \]

Therefore, the mixed strategy \((1/3, 2/3)\) for Alice makes Bob indifferent between his two actions, if Bob has type U.

If Bob is interested in Alice, his payoff for playing C is:

\[ 5p + 0(1-p) = 5p. \]  
\[ (55) \]

His payoff for playing H is

\[ 0p + 10(1-p) = 10(1-p). \]  
\[ (56) \]

Plugging in Alice’s strategy yields a payoff of 5 \( \left( \frac{1}{3} \right) = \frac{5}{3} \) for playing C, and 10 \( \left( \frac{2}{3} \right) = \frac{20}{3} \) for playing H. Bob’s payoff if he has type I is strictly greater when playing H, so \( q_I = 0 \).

Alice’s payoff Alice for playing C, when \( q_I = 0 \), is:

\[
\Pr(I) \left[ 10q_I + 0(1-q_I) \right] + \Pr(U) \left[ 10q_U + 0(1-q_U) \right]
\]
\[
= \frac{1}{2} \left[ 10q_I \right] + \frac{1}{2} \left[ 10q_U \right]
\]
\[
= \frac{1}{2} \left[ 10(0) \right] + \frac{1}{2} \left[ 10q_U \right]
\]
\[
= 5q_U.
\]
\[(57)-(60)\]

Her payoff for playing H, when \( q_I = 0 \), is:

\[
\Pr(I) \left[ 0q_I + 5(1-q_I) \right] + \Pr(U) \left[ 0q_U + 5(1-q_U) \right]
\]
\[
= \frac{1}{2} \left[ 5(1-q_I) \right] + \frac{1}{2} \left[ 5(1-q_U) \right]
\]
\[
= \frac{1}{2} \left[ 5(1) \right] + \frac{1}{2} \left[ 5(1-q_U) \right]
\]
\[
= \frac{5}{2} + \frac{5}{2}(1-q_U).
\]
\[(61)-(64)\]
Equating the payoffs, we solve for $q_U$:

$$5q_U = \frac{5}{2} + \frac{5}{2}(1 - q_U) \quad (65)$$

$$10q_U = 5 + 5(1 - q_U) \quad (66)$$

$$15q_U = 10 \quad (67)$$

$$q_U = \frac{2}{3} \quad (68)$$

Putting it all together:

1. Alice plays C with probability $p = \frac{1}{3}$, and H with probability $1 - p = \frac{2}{3}$.
2. If Bob has type I, then he plays C with probability $q_I = 0$, and H with probability $1 - q_I = 1$.
3. If Bob has type U, then he plays C with probability $q_U = \frac{2}{3}$, and H with probability $1 - q_U = \frac{1}{3}$.

We can describe the mixed strategy compactly, as follows:

$$\begin{bmatrix}
(\frac{1}{3}, \frac{2}{3}) \\
(\frac{2}{3}, \frac{1}{3})
\end{bmatrix}
\begin{bmatrix}
(0, 1) \\
(\frac{2}{3}, \frac{1}{3})
\end{bmatrix}
\begin{bmatrix}
\frac{1}{3}, \frac{2}{3} \\
\frac{2}{3}, \frac{1}{3}
\end{bmatrix}$$

**Verification** We now verify that this mixed strategy is in fact a Bayes-Nash equilibrium. Fixing Alice’s (Bob’s) strategy, it should be the case that Bob (Alice) cannot employ an alternative mixed strategy that yields strictly more utility.

**Alice** The expected utility Alice receives for playing C is

$$u_a(C) = \Pr(I) [10 (q_I) + 0 (1 - q_I)] + \Pr(U) [10 (q_U) + 0 (1 - q_U)]$$

$$= \frac{1}{2} [10 (0) + 0 (1)] + \frac{1}{2} \left[ 10 \left( \frac{2}{3} \right) + 0 \left( \frac{1}{3} \right) \right]$$

$$= \frac{10}{3} \quad (71)$$

The expected utility Alice receives for playing H is

$$u_a(H) = \Pr(I) [0 (q_I) + 5 (1 - q_I)] + \Pr(U) [0 (q_U) + 5 (1 - q_U)]$$

$$= \frac{1}{2} [0 (0) + 5 (1)] + \frac{1}{2} \left[ 0 \left( \frac{2}{3} \right) + 5 \left( \frac{1}{3} \right) \right]$$

$$= \frac{10}{3} \quad (74)$$
\[ = \frac{10}{3}. \quad (75) \]

Since the expected utilities are equal, Alice is indifferent between playing C and H, and cannot improve her expected utility by mixing.

**Bob (type I)** The expected utility Bob (type I) receives for playing C is

\[
u_b(C) = 5(p) + 0(1 - p)
\]
\[= 5p \quad (76)\]
\[= 5 \quad (77)\]
\[= \frac{5}{3}. \quad (78)\]

The expected utility Bob (type I) receives for playing H is

\[
u_b(H) = 0(p) + 10(1 - p)
\]
\[= 10(1 - p) \quad (79)\]
\[= \frac{20}{3}. \quad (80)\]

Since the expected utility of playing H is strictly larger than the expected utility of playing C, Bob will play H.

**Bob (type U)** The expected utility Bob (type U) receives for playing C is

\[
u_b(C) = 0(p) + 5(1 - p)
\]
\[= 5(1 - p) \quad (82)\]
\[= \frac{10}{3}. \quad (83)\]

The expected utility Bob (type U) receives for playing H is

\[
u_b(H) = 10(p) + 0(1 - p)
\]
\[= 10(p) \quad (85)\]
\[= \frac{10}{3}. \quad (86)\]

Since the expected utilities are equal, Bob is indifferent between playing C and H, and cannot improve his expected utility by mixing.