Battle of the Sexes
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We present an example of a (complete-information) normal-form games, and calculate its Nash equilibria.

1 Battle of the Sexes

We have a couple, Alice and Bob, who are planning to attend an event together this evening. Unfortunately, both are forgetful, and can’t recall which of two possible events they had planned to attend. To make matters worse, they cannot communicate with each other before the event.\(^1\) While each person prefers one event over the other, both Alice and Bob derive no happiness if they don’t attend the same event. Payoffs for all possible outcomes are given in Figure 1.

<table>
<thead>
<tr>
<th></th>
<th>Concert</th>
<th>Lecture</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concert</td>
<td>7, 3</td>
<td>0, 0</td>
</tr>
<tr>
<td>Lecture</td>
<td>0, 0</td>
<td>3, 7</td>
</tr>
</tbody>
</table>

\(^1\) This thought experiment was invented before the proliferation of cellular technology.

Figure 1: The payoff matrix describing the payoffs Alice and Bob receive for attending one of two possible events. Alice is the row player. Bob is the column player. If Alice and Bob both attend the Concert, Alice receives payoff 7, and Bob receives payoff 3.

Problem

1. Draw the best-response correspondences for each player on the same graph.
2. Label all Nash equilibria that involve pure strategies.
3. Label all Nash equilibria that involve mixed strategies.
4. What are the players’ expected payoffs at each of these Nash equilibria?

Solution

We use the following notation:

- \(p\): Probability Alice goes to the concert.
- \(q\): Probability Bob goes to the concert.
- \(u_x\): utility of \(x \in \{\text{Alice, Bob}\}\), where \(A\) stands for Alice, and \(B\) stands for Bob.
1.1 Alice

Alice prefers the concert when her expected utility of the concert action exceeds that of the lecture action. She prefers the lecture when her expected utility of the lecture action exceeds that of the concert action. When these two expected utilities are equal, she is indifferent between her two pure strategies, and willing to mix. Let’s calculate these expected utilities.

Alice’s expected utility of going to the concert is:

\[
E_q[u_A(\text{concert}, \cdot)] = qu_A(\text{concert, concert}) + (1 - q)u_A(\text{concert, lecture}) = qu_A(\text{concert, concert}).
\] (1)

Alice’s expected utility of going to the lecture is:

\[
E_q[u_A(\text{lecture, \cdot})] = qu_A(\text{lecture, concert}) + (1 - q)u_A(\text{lecture, lecture}) = (1 - q)u_A(\text{lecture, lecture}).
\] (2)

Alice’s total expected utility is:

\[
E[u_A(\cdot, \cdot)] = pE[u_A(\text{concert, \cdot})] + (1 - p)E[u_A(\text{lecture, \cdot})] = pqu_A(\text{concert, concert}) + (1 - p)(1 - q)u_A(\text{lecture, concert})
\] (3)

If \(qu_A(\text{concert, concert}) > (1 - q)u_A(\text{lecture, lecture})\), then \(p = 1\) is optimal. If \(qu_A(\text{concert, concert}) < (1 - q)u_A(\text{lecture, lecture})\), then \(p = 0\) is optimal. Otherwise, \(p \in [0, 1]\) To summarize:

\[
p \begin{cases} 
  1, & \text{if } qu_A(\text{concert, concert}) > (1 - q)u_A(\text{lecture, lecture}) \\
  \in [0, 1], & \text{if } qu_A(\text{concert, concert}) = (1 - q)u_A(\text{lecture, lecture}) \\
  0, & \text{otherwise}.
\end{cases}
\] (4)

Plugging in the numbers from the payoff matrix, we have:

\[
qu_A(\text{concert, concert}) = (1 - q)u_A(\text{lecture, lecture})
\] (5)
\[
q \cdot 7 = (1 - q) \cdot 3
\] (6)
\[
7q = 3 - 3q
\] (7)
\[
10q = 3
\] (8)
\[
q = \frac{3}{10}
\] (9)

Hence,

\[
p \begin{cases} 
  1, & \text{if } q > \frac{3}{10} \\
  \in [0, 1], & \text{if } q = \frac{3}{10} \\
  0, & \text{otherwise}.
\end{cases}
\] (10)
Alice’s **best-response correspondence** is depicted in Figure 2. Note that this correspondence is *not* a function.

![Figure 2: Alice](image)

### 1.2 Bob

Like Alice, Bob prefers the concert when his expected utility of the concert action exceeds that of the lecture action. He prefers the lecture when his expected utility of the lecture action exceeds that of the concert action. When these two expected utilities are equal, he is indifferent between his two pure strategies, and willing to mix. Let’s calculate these expected utilities.

Bob’s expected utility of going to the concert is:

\[
E_p [u_B(\text{concert}, \cdot)] = pu_B(\text{concert}, \text{concert}) + (1 - p)u_B(\text{concert}, \text{lecture})
\]

\[= pu_B(\text{concert}, \text{concert}). \tag{14}\]

Bob’s expected utility of going to the lecture is:

\[
E_p [u_B(\text{lecture}, \cdot)] = pu_B(\text{lecture}, \text{concert}) + (1 - p)u_B(\text{lecture}, \text{lecture})
\]

\[= (1 - p)u_B(\text{lecture}, \text{lecture}). \tag{16}\]

Bob’s total expected utility is:

\[
E [u_B(\cdot, \cdot)] = qE [u_B(\text{concert}, \cdot)] + (1 - q)E [u_B(\text{lecture}, \cdot)]
\]

\[= qpu_B(\text{concert}, \text{concert}) + (1 - q)(1 - p)u_B(\text{lecture}, \text{lecture}). \tag{19}\]
If $pu_B(\text{concert, concert}) > (1 - p)u_B(\text{lecture, lecture})$, then $q = 1$ is optimal. If $pu_B(\text{concert, concert}) < (1 - p)u_B(\text{lecture, lecture})$, then $q = 0$ is optimal. Otherwise, $q \in [0, 1]$ To summarize:

$$
p \begin{cases} 
1, & \text{if } pu_B(\text{concert, concert}) > (1 - p)u_B(\text{lecture, lecture}) \\
\in [0, 1], & \text{if } pu_B(\text{concert, concert}) = (1 - p)u_B(\text{lecture, lecture}) \\
0, & \text{otherwise.}
\end{cases}
$$

(20)

Plugging in the numbers from the payoff matrix, we have:

$$
p u_B(\text{concert, concert}) = (1 - p)u_B(\text{lecture, lecture})
$$

(21)

$$
p \cdot 3 = (1 - p) \cdot 7
$$

(22)

$$
3p = 7 - 7p
$$

(23)

$$
10p = 7
$$

(24)

$$
p = \frac{7}{10}
$$

(25)

Hence,

$$
q \begin{cases} 
1, & \text{if } p > \frac{7}{10} \\
\in [0, 1], & \text{if } p = \frac{7}{10} \\
0, & \text{otherwise.}
\end{cases}
$$

(26)

Bob’s best-response correspondence is depicted in Figure 3. Note that this correspondence is not a function.

1.3 Nash Equilibria

Figure 4 plots the two curves together. From these overlapping plots, we can visualize the equilibrium solutions. They occur at the inter-
sections of the two best-respondence correspondences.

1.4 Utility

We now compute players’ expected utilities at the mixed strategy Nash equilibrium. Alice’s expected utility is:

$$E[u_A(\cdot, \cdot)] = p E[u_A(\text{concert}, \cdot)] + (1 - p) E[u_A(\text{lecture}, \cdot)]$$

$$= \frac{7}{10} \cdot \frac{3}{10} \cdot 7 + \frac{3}{10} \cdot \frac{7}{10} \cdot 3$$

$$= \frac{21}{10}.$$ 

Since the game is symmetric, it is also the case that Bob’s expected utility $$E[u_B(\cdot, \cdot)] = \frac{21}{10}.$$