Due Date: Tuesday, April 17, 2018, 9:00 PM.

We encourage you to work in groups of size two. Each group need only submit one solution. Your submission must be typeset using LaTeX. Please submit a physical copy of your solutions to the handin bin on the second floor.

Regardless of your enrollment level, you must solve all four problems.¹

Unit-demand Bidders

Throughout this assignment, we assume the following multi-parameter auction setting:

1. There is a set \( G \) of \( m \) distinct goods.

2. There is a set \( N \) of \( n \) bidders, with each bidder \( i \) characterized by a private valuation \( v_i \) that ascribes value \( v_{ij} \) to good \( j \).

3. Each bidder \( i \) is further characterized by a unit-demand valuation, meaning bidder \( i \) values a bundle \( X \subseteq G \) as \( v_i(X) = \max_{j \in X} v_{ij} \).

An outcome in this model consists of an allocation and payment scheme (also called a pricing). An allocation is a matching \( M \) of goods to bidders, in which each bidder is matched to at most one good, and each good, to at most one bidder. A pricing is a vector of prices \( q \in \mathbb{R}^m_+ \), where \( q(j) \) is the price of good \( j \in G \). We denote the good matched to bidder \( i \) under \( M \) by \( M(i) \in G \), and its price by \( q(M(i)) \in \mathbb{R}_+ \). A bidder \( i \) might not be matched in \( M \), in which case we define \( M(i) = \emptyset \) and \( v_i(\emptyset) = q(\emptyset) = 0 \). The welfare of a matching \( M \) is defined as \( W(M) = \sum_i v_i(M(i)) \). Finally, we define a Walrasian equilibrium (WE) \((M, q)\) as follows:

\( WE1 \) Each bidder \( i \) is allocated a preferred good: i.e., one such that

\[
    v_i(M(i)) - q(M(i)) \geq v_i(j) - q(j), \quad \forall j \in G.
\]

Note that this condition implies that all bidders’ utilities are non-negative, since \( M(i) = \emptyset \) is a valid allocation for bidder \( i \).

\( WE2 \) The market clears: i.e., if good \( j \) is unallocated, then \( q(j) = 0 \).

¹ This is not a typo! There are no grad-only problems this week.
1 Walras’ Law

Walras’ Law\(^2\) asserts that, at equilibrium, the excess demand of a market must sum to zero, or equivalently, the excess supply of a market must sum to zero. In a market with unit-demand bidders, the total value of supply is \(\sum_{j \in G} q(j)\) and the total value of demand is \(\sum_{i \in N} q(M(i))\). The excess demand (or supply) is the difference between these two quantities. Formally state Walras’ Law, and prove its equivalence to WE\(^2\).

2 Mixing and Matching Matchings

Prove the following claim: if \((M, q)\) is a Walrasian equilibrium and \(M^* \neq M\) is a welfare-maximizing matching, then \((M^*, q)\) is a Walrasian equilibrium. \textbf{Hint:} Use the First Welfare Theorem.

3 Walrasian Equilibrium Prices form a Lattice

Prove that Walrasian Equilibrium price vectors form a lattice. Concretely, let \((M, q^1)\) and \((M, q^2)\) be two Walrasian equilibria outcomes. Show that \((M, q^1 \land q^2)\) and \((M, q^1 \lor q^2)\) are also Walrasian equilibria, where \(q^1 \land q^2\) and \(q^1 \lor q^2\) are the price vectors obtained by taking the component-wise minimum and maximum of \(q^1\) and \(q^2\), respectively: i.e.,

- \((q^1 \land q^2)(j) = \min\{q_1(j), q_2(j)\}\)
- \((q^1 \lor q^2)(j) = \max\{q_1(j), q_2(j)\}\)

4 \(\epsilon\)-Walrasian Equilibrium and the First Welfare Theorem

Define an \(\epsilon\)-Walrasian Equilibrium \((M, q)\) as follow:

\(\text{WE}_1\) Each bidder \(i\) is allocated a preferred good: i.e., one such that

\[v_i(M(i)) - q(M(i)) \geq v_i(j) - q(j) - \epsilon, \quad \forall j \in G,\]

and all bidders’ utilities are non-negative.

\(\text{WE}_2\) The market clears: i.e., if good \(j\) is unallocated, then \(q(j) = 0\).

Prove the following approximate version of the celebrated First Welfare Theorem: if \((M, q)\) is an \(\epsilon\)-Walrasian equilibrium, then \(W(M)\) is within \(\epsilon \min\{n, m\}\) of the value of a welfare-maximizing matching.