1 Bayesian Constraints, Revisited

Recall the Bayesian (interim) constraints on Problem 3 on Homework 3. Let $P$ be the set of auctions that satisfy IC, IR, and ex-post feasibility. Let $I$ be the set of auctions that satisfy BIC, IIR, and ex-post feasibility.

Notice that since BIC and IIR are weaker conditions than IC and IR, $P$ is a subset of $I$, and thus,

$$\max_{a \in P} EW(a) \leq \max_{a \in I} EW(a)$$

where $EW(a)$ is the expected welfare achieved by auction $a$.

Show that, in fact,

$$\max_{a \in P} EW(a) = \max_{a \in I} EW(a)$$

N.B. This same result holds for revenue: expected revenue is also the same regardless of whether the IC and IR constraints are interim or ex-post. You need not prove this result; though you are welcome to. It is not significantly harder than the proof for welfare.

2 Another Auction with Reserve

Reserve prices are necessary to maximize expected revenue, since they given auctioneers the flexibility to charge more money to bidders whom they expect to have high values, while still preserving incentive compatibility and individual rationality.

Bob understands this, but he does not understand why the revenue-maximizing single-good auction allocates the good to the bidder with the highest virtual value, rather than the bidder with the highest value. To him, it seems like the revenue-maximizing auction should
allocate to the bidder with the highest value, since that bidder is willing to pay the most! So Bob proposes the following revision to Myerson’s revenue-maximizing single-good auction: use the same reserve prices as before, but allocate to the bidder with the highest value among the bidders who bid above their reserve prices.

To be clear, Bob’s allocation rule works as follows:

- Assume that \( v_i \sim F_i \) for all bidders \( i \), and assume that \( F_i \) has bounded support and is regular.
- Determine the set \( C \) of bidders \( i \) who bid at least \( \phi_i^{-1}(0) \), where \( \phi_i \) is bidder \( i \)’s virtual value function.
- Allocate the good to the highest bidder in \( C \).

1. Derive the payment rule that creates an auction that satisfies IC and IR with Bob’s allocation rule. Prove that Bob’s allocation rule together with this payment rule satisfies IC and IR.

2. When 0 or 1 bidders bid above their reserve prices, Bob’s and Myerson’s auctions yield the exact same outcome.

   Consider the case in which there are at least two bidders who bid above their reserve prices, and among these bidders, the bidder with the highest value is not the same as the bidder with the highest virtual value. Prove that Bob’s auction yields weakly greater expected revenue than Myerson’s auction in this case.

3. Consider the other case, in which the bidder with the highest value also has the highest virtual value. In this case:

   (a) Provide an example in which Bob’s auction produces greater revenue.

   (b) Provide an example in which Myerson’s auction produces greater revenue.

   Your examples should define the number of bidders, each bidder’s value distribution, and each bidder’s value.

4. Prove that if all bidders have the same value distribution, and their (shared) virtual value function is strictly increasing, Bob’s and Myerson’s auctions are exactly the same (the same bidder wins, and they pay the same price).

5. Suppose that there are just two bidders. Bidder A whose value is drawn from \( U(0, 1) \) and Bidder B whose value is drawn from \( U(0, 2) \). Compute the expected revenue of both auctions.

   \textbf{Hint:} Setup cases based on the bidders’ values and virtual values. You can apply previous results to several of the cases.
3 Bulow-Klemperer

Consider a single-item auction with \( n \) bidders with valuations drawn \( i.i.d. \) from a regular distribution \( F \). Prove that \( V_n \), the expected revenue of the Vickrey auction (2nd price with no reserve), is at least \( \frac{n-1}{n} \) times that of the optimal auction (with the same number of bidders). You may assume that \( V_n \) is increasing and concave in \( n \), \( \forall n \geq 1 \). In other words, \( \forall a, b, t \in \mathbb{Z}^+ \) such that \( t = \lambda a + (1 - \lambda) b \) for \( 0 \leq \lambda \leq 1 \) we have that \( V_t \geq \lambda V_a + (1 - \lambda) V_b \).

4 Treating Irregular Distributions

A regular distribution \( F \) yields a monotone non-decreasing virtual value function \( \varphi \), and a concave revenue curve \( R \). If \( F \) is not regular, then it is not guaranteed that the virtual value function is non-decreasing, and that the revenue curve is concave. In this setting, maximizing virtual surplus may not yield a monotone non-decreasing allocation function. In this problem, we will see how one can deal with distributions that are not regular.

Suppose we have an allocation function \( \hat{x}(v(q)) \). Let \( \hat{y}(q(v)) = \hat{x}(v(q)) \).

1. Show that total expected virtual surplus can be written as

\[
\mathbb{E}_{v \sim F} [\varphi(v) \hat{x}(v)] = \mathbb{E}_{q \sim \mathcal{U}(0,1)} \left[ \frac{dR(q)}{dq} \hat{y}(q) \right].
\]  

2. Show that the total expected virtual surplus can be written as

\[
\mathbb{E}_{q \sim \mathcal{U}(0,1)} \left[ \frac{dR(q)}{dq} \hat{y}(q) \right] = \mathbb{E}_{q \sim \mathcal{U}(0,1)} \left[ -R(q) \frac{d\hat{y}(q)}{dq} \right].
\]  

3. Let \( \bar{R} \) be the smallest concave upper-bound of \( R \). Prove that if \( \hat{y}(q) \) is monotone non-increasing with respect to \( q \), then

\[
\mathbb{E}_{q \sim \mathcal{U}(0,1)} \left[ -R(q) \frac{d\hat{y}(q)}{dq} \right] \leq \mathbb{E}_{q \sim \mathcal{U}(0,1)} \left[ -\bar{R}(q) \frac{d\hat{y}(q)}{dq} \right].
\]  

What this means is that by using the ironed revenue curve \( \bar{R}(q) \), we can construct an ironed virtual value function

\[
\bar{\varphi}(v(q)) = \frac{d\bar{R}(q)}{dq},
\]  

and maximize pointwise, using ironed virtual values. This will yield a monotone allocation rule that gets at least as much virtual welfare as that of the original, un-ironed virtual values.