Due Date: Tuesday, February 28, 2017. 9:00 PM.

We encourage you to work in groups of size two. Each group need only submit one solution. Your submission must be typeset using \LaTeX. You must submit both the \LaTeX source and the rendered PDF. \LaTeX files without an accompanying PDF will not be graded. Likewise, PDF files without accompanying \LaTeX source will not be graded. Use the handin script to submit.

For 1000-level credit, you need only solve the first four problems. For 2000-level credit, you must solve six.

1  The Revelation Principle, Revisited

We (re)define the English auction as follows:

On round $k = 1, 2, \ldots, t$, the auctioneer offers the good at price $ke$, asking all bidders if they are interested in the good at that price, in which case they become part of the publicly visible set $S_k$ of bidders interested in the good at round $k$. The activity rule is such that bidders may choose to enter and exit the auction as they please: thus, it is not necessarily the case that $S_{k+1} \subseteq S_k$. The auction terminates at round $t$ when fewer than two bidders remain interested in the good. If there is one interested bidder, then she wins, paying $te$. If there are no interested bidders, then a winner is selected uniformly at random from the set $S_{t-1}$; this bidder pays $(t-1)e$ for the good.

We have seen in class that this mechanism is not DSIC due to some bizarre strategies for which sincere bidding is not the best response. Additionally, there are some edge cases where it is possible to bid insincerely and obtain a slightly greater (by no more than $e$) payoff than sincere bidding. ¹ We can use the Revelation Principle to produce a mechanism that eliminates the first of these two objections. Specifically, you will show that the mechanism $M'$ resulting from applying the Revelation Principle to the English Auction is DSIC up to $e$. What we mean by this is that regardless of other players’ bids and strategies, no bidder can improve on the payoff obtained by sincere bidding by more than $e$.

1. Prove that $M'$ is DSIC, up to $e$.

2. Describe in a sentence (or two) why it is reasonable that the loopholes in the English auction are closed as a result of applying the

¹ Many of you found these in the previous homework. If you did not, we encourage you to think about this.

² That is, for every bidder $i$, $M'$ will accept a bid $b_i$, answer queries consistently according to the valuation $b_i$, then produce the same outcome as in the original mechanism.
Revelation Principle.

Now suppose an auctioneer named Dave is selling \( m \) distinct goods. Suppose further that, for each bidder, the values of these goods are independent: there are no complements\(^3\) or substitutes\(^4\). Dave decides to auction the goods off simultaneously by running one giant auction that consists of \( m \) parallel English auctions. On round \( k \), Dave keeps track of the set \( S_{k,j} \) of bidders interested in good \( j \) at the price \( k \epsilon \). Each good \( j \) is awarded when \( |S_{t,j}| \leq 1 \), as in a single English auction, and the payments are the same as well.

A single bidder can win any number of goods. You can imagine that goods are allocated immediately or at the very end of the giant auction; it doesn’t matter, since bidders values for goods are independent of their other holdings.

There’s just one catch: Dave is smart, so he enforces an activity rule, namely that \( S_{k+1,j} \subseteq S_{k,j} \), for every \( j, k \), in attempt to close the loophole that prevented the English Auction from being DSIC.

3. Prove (by example) that Dave’s auction is not DSIC (even up to \( \epsilon \)).

4. Prove that if you apply the Revelation Principle to the sincere-bidding BNE\(^5\) in Dave’s auction, the result is a DSIC mechanism, up to \( \epsilon \).

2 Collusion in the Second-Price Auction

Recall that if an auction is DSIC (dominant strategy incentive compatible), then no single bidder has an incentive to deviate from bidding truthfully. But the story is not quite so simple if bidders can collude, meaning submit false bids in coordinated way.

Consider a Vickrey (i.e., second-price) auction in which all but a subset \( S \) of the bidders bid truthfully. Instead, the members of \( S \) attempt to collude to increase their collective payoff. Prove necessary and sufficient conditions on the set \( S \) (in terms of the bidders’ valuations) such that the bidders in \( S \) can increase their collective payoff via non-truthful bidding.

3 Sponsored Search Extension

In this problem, we generalize the sponsored search problem slightly to include a publicly known quality \( \beta_i \) for each bidder \( i \). Intuitively, we can view \( \alpha_j \) as the probability a user will see the advertisement, independent of what it looks like; and \( \beta_i \) is the probability that a user will click on the advertisement given that he sees it. Therefore, if bidder \( i \) is placed in slot \( j \), her payoff is given by \( \alpha_j \beta_i v_i \) (minus payment).

\(^{3}\) For example, a kayak and a paddle, which complement one another, and hence, are worth more in conjunction.
\(^{4}\) Goods that are worth less in conjunction, such as two competing brands of kayaks or two competing brands of paddles.

\(^{5}\) Once again, you may assume without proof that sincere bidding is a BNE, up to \( \epsilon \).
1. Describe an allocation rule which maximizes the social welfare. Justify your answer.

2. Argue that the allocation rule is monotone and use Myerson’s formula to produce a payment rule that, together with 1, forms a DSIC mechanism.

4 Bayesian Constraints

Suppose that, rather than insisting on incentive compatibility to hold always, and for individual rationality hold always, we use constraints such that they hold in expectation. Let the interim allocation and interim payment functions be

\[
\hat{x}_i(v_i) = \mathbb{E}_{v_{-i} \sim F_{-i}} [x_i(v_i, v_{-i})], \quad \forall i \in N, \forall v_i \in T_i,
\]

\[
\hat{p}_i(v_i) = \mathbb{E}_{v_{-i} \sim F_{-i}} [p_i(v_i, v_{-i})], \quad \forall i \in N, \forall v_i \in T_i,
\]

respectively. Bayesian incentive compatibility (BIC) tells us that bidding truthfully is, in expectation, utility maximizing:

\[
v_i \hat{x}_i(v_i) - \hat{p}_i(v_i) \geq v_i \hat{x}_i(t_i) - \hat{p}_i(t_i), \quad \forall i \in N, \forall v_i, t_i \in T_i.
\]

Interim individual rationality (IIR) tells us that bidding truthfully, in expectation, leads to non-negative utility:

\[
v_i \hat{x}_i(v_i) - \hat{p}_i(v_i) \geq 0, \quad \forall i \in N, \forall v_i, t_i \in T_i.
\]

Myerson’s analysis tells us that:

1. interim allocations must be monotone non-decreasing:

\[
\hat{x}_i(v_i) \geq \hat{x}_i(t_i), \quad \forall i \in N, \forall v_i \geq t_i \in T.
\]

2. payments must take the following form:

\[
\hat{p}_i(v_i) = v_i \hat{x}_i(v_i) - \int_0^{v_i} \hat{x}_i(z) \, dz, \quad \forall i \in N, \forall v_i \geq t_i \in T
\]

Suppose that for each bidder $i \in N$, valuations $v_i$ are independent draws from distribution $F_i$:

\[
v_i \sim F_i, \quad \forall i \in N.
\]

1. The interim allocation at $v_i$ is the probability of winning with type $v_i$. If there are $n = 2$ bidders, each drawing valuation from a $U(0, 1)$ distribution, what should the interim allocation look like?
2. With what you have derived analytically in the first problem, what is the analytic form of the interim payment formula? (Ie, what is the payment formula when there are two bidders, each drawing valuation from a \( U(0, 1) \) distribution)?

3. Generalize your result from the first problem. Given a set of bidders \( N = \{1, \ldots, n\} \), their valuation distributions \( F_i \) and types \( v_{ij} \), give the analytic form the interim allocation function must take.

4. Show that the maximum possible total welfare you can generate using the IC and IR constraints is equal to the maximum possible total welfare you can generate had BIC and IIR been used.

5 \textbf{The Knapsack Auction}

The \textit{knapsack problem} is a famous NP-hard\(^6\) problem in combinatorial optimization. The problem is stated as follows:

There are \( n \) items; each item \( i \) has a some weight \( w_i \) and value \( v_i \). The knapsack has a total capacity \( W \); the goal is to find some subset \( S \) of items of maximal total value with total weight less than \( W \). Written as a linear program,

\[
\max \sum_{i=1}^{n} x_i v_i \\
\text{subject to} \sum_{i=1}^{n} x_i w_i \leq W, x \in \{0, 1\}.
\]

We can frame this problem as a mechanism design problem as follows. Each bidder has an item that they would like to put in the knapsack. Each item has a public parameter \( w_i \) and a private value \( v_i \). An auction will take place, after which some set \( S \) of bidders will place their items (of total weight less than \( W \)) in the knapsack and pay some amount of money to the auctioneer. One possible application of this is auctioning off commercial snippets in a 5-minute ad break\(^7\).

Since the problem is NP-hard, we cannot hope to find a polynomial-time welfare-maximizing solution. Instead, you will produce a polynomial-time, DSIC mechanism that is a \( 2 \)-approximation\(^8\) of the optimal welfare.

We propose the following greedy allocation scheme: Sort the bidders’ items by their ratios \( \frac{v_i}{w_i} \) and allocate items in that order until there is no room left in the knapsack.

1. Show that this allocation scheme is not a \( 2 \)-approximation by producing an example where it fails to achieve 50% of the optimal welfare.

\(^6\) There are no known polynomial-time solutions.

\(^7\) Here, the weight is the time of the commercial in seconds.

\(^8\) Meaning that for any set possible set of valuations and weights, we always achieve at least 50% of the optimal welfare.
2. Propose a (small) improvement to the allocation scheme and prove that it is now a 2-approximation of the optimal welfare\(^9\).

3. Argue that the allocation scheme is monotone and use Myerson’s payment formula to produce payments such that the resulting mechanism is DSIC.

6 Allocation Discontinuity

We saw in class that incentive compatibility and individual rationality led us to two properties:

1. Allocation monotonicity: the probability of being allocated should not decrease as one’s bid increases.

2. Payment formula: payments are given by the following equation:

\[
p_i(v_i, v_{-i}) = v_i x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(z, v_{-i}) \, dz, \quad \forall i \in N, \forall v_i \in T_i, \forall v_{-i} \in T_{-i}.
\]

(8)

1. In a second-price auction, the allocation function is piecewise continuous and non-increasing on any continuous interval. Let \( B \) be the second highest bid, and suppose all bids are unique. Then:

\[
x_i(v_i, v_{-i}) = \begin{cases} 1, & \text{if } v_i > B \\ 1/2, & \text{if } v_i = B \\ 0, & \text{otherwise.} \end{cases}
\]

(9)

In this case, the allocation function is a Heaviside step function with discontinuity at \( v_i = B \). The payment formula tells us that if \( i \) wins, she should pay

\[
p_i(v_i, v_{-i}) = v_i x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(z, v_{-i}) \, dz
\]

\[
= v_i 1 - \left( \int_0^B 0 \, dz + \int_B^{v_i} 1 \, dz \right)
\]

\[
= v_i 1 - (0 + v_i - B)
\]

\[
= B.
\]

(10-13)

Another way to express this is

\[
p_i(v_i, v_{-i}) = B \cdot [\text{jump in } x_i \text{ at } B].
\]

(14)

1. Suppose that the allocation function is piecewise continuous, non-increasing on any continuous interval, and discontinuous at points \( \{z_1, z_2, \ldots, z_\ell\} \), where these \( \ell \) points are the points of discontinuity in the allocation function up to some bid \( v_i \). That is, there are

\* Use the notation APX and OPT and prove a bound on their ratio, similar to what we did in the Posted Price Mechanism (you can find the notes for this under ‘Lectures’ on the website).
exactly \( \ell \) points in the allocation where there are jumps from 0 up to \( v_i \). Given that the allocation rule is monotone non-decreasing, show that the payment rule can be expressed as

\[
p_i(v_i, v_{-i}) = \sum_{j=1}^{\ell} z_j \cdot [\text{jump in } x_i(\cdot, v_{-i}) \text{ at } z_j].
\]  \hspace{1cm} (15)

In particular, show this in two ways:

(a) Analytically show that the equation must hold.

(b) Proof by picture.

\[\text{In the second-price auction described above, there is one jump in the allocation function. Two if you include ties (i.e., an allocation probability of 1/2 if } i \text{ bids as much as the maximum bid over the other bidders.)}\]