Due Date: Tuesday, February 27, 2018. 9:00 PM.

We encourage you to work in groups of size two. Each group need only submit one solution. Your submission must be typeset using \LaTeX. Submit to the handin bin in the CIT.

For 1000-level credit, you need only solve the first three problems. For 2000-level credit, you must solve the first four problems. The fifth problem can be completed by anyone for extra credit.

1 Ascending Auctions

In this problem, we introduce a new type of auction called the open outcry, ascending, a.k.a the English, auction. Rather than issue a single query for sealed bids, this auction consists of a number of rounds. On round $k = 1, 2, \ldots, t$, the auctioneer offers the good at price $ke$, for some small $e > 0$, asking all bidders if they are interested in the good at that price, in which case they become part of the publicly visible set $S_k$ of bidders interested in the good at round $k$. Bidders may choose to enter and exit the auction as they please: thus, it is not necessarily the case that $S_{k+1} \subseteq S_k$. The auction continues so long as more than one bidder is interested. The auction terminates, say at round $t$, when one or fewer bidders remain interested. If there is one interested bidder at round $t$, then she wins, paying $te$; if there are no interested bidders, then a winner is selected at random from the set of interested bidders during round $t-1$. This winner pays $(t-1)e$.

This is our first example of an indirect mechanism. A mechanism is direct if it is sealed-bid, and if reports to the center (i.e., actions) must take the form of types; that is, players are asked to reveal all their private information up front. In contrast, in an indirect mechanism, the players' private information is revealed gradually, if at all; for example, via the repeated binary queries, like “Would you like the item at price $p$?” in an English auction.

In addition to providing incentive compatibility for free, the Revelation Principle can also transform an indirect mechanism into a direct one. This is useful because it tells us that, for a given model (i.e., a set of players, actions, and types), we need not concern ourselves with the unlimited number of imaginable indirect mechanisms, and their corresponding unlimited number of equilibria. On the con-
try, it suffices to study direct mechanisms and their equilibria, since an equilibrium in any indirect mechanism can be transformed into a corresponding equilibrium in a direct mechanism.

1. Bob notices that an English auction is very similar to a second-price auction, since the winner pays the essentially second-highest bidder’s bid: i.e., the price right before the second-highest bidder drops out. Since the second-price auction is DSIC, Bob believes that the English auction must be DSIC, in some sense, too.

To state Bob’s intuition more formally, we must define what it means for an indirect auction to be DSIC. Let’s start by defining an analog of truthful bidding for indirect auctions, namely sincere bidding. A sincere bidder responds to all queries immediately, and truthfully, meaning in a way that is consistent with their values: e.g., answering “yes” to the query “Would you like the item at $10?” iff their value is greater than $10. An indirect auction is DSIC iff sincere bidding is a dominant strategy, for all bidders.

Show that sincere bidding is not DSIC in the English Auction by showing that sincere bidding is not even a Bayes-Nash equilibrium. Construct a counterexample by finding values for the bidders in which a bidder can obtain slightly more ($\epsilon$) expected utility by deviating from sincere bidding.

2. We say that an indirect auction is DSIC up to $\epsilon$ iff no bidder who deviates from sincere bidding can improve upon sincere bidding by more than $\epsilon$. Likewise, we can define the notion of any equilibrium “up to $\epsilon$,” where deviating from the equilibrium strategy can not improve expected utility by more than $\epsilon$, assuming the other players are conforming.

Prove that sincere bidding in the English Auction is a Bayes-Nash Equilibrium, up to $\epsilon$.

3. Show by counterexample that the English auction is not DSIC, even up to $\epsilon$.

Keep in mind that the other bidders need not bid sincerely, and that strategies in the English Auction can be entirely bizarre: their behavior need not be consistent from round to round.

4.(a) Apply the Revelation Principle to transform the English auction, in which sincere bidding is an $\epsilon$-BNE, into a direct mechanism, which is DSIC, up to $\epsilon$.

(b) Describe in a sentence (or two) why it is reasonable that applying the Revelation Principle closes the loopholes in the English auction.
2 A Revenue-Maximizing Auction

Recall the revenue-maximizing allocation scheme for the single-good auction: allocate the good to a bidder with the highest virtual value, among the bidders with nonnegative virtual values. (N.B. If all of the bidders’ virtual values are negative, then the good is not allocated.)

Your goal in this problem is to use this allocation rule to create an auction that satisfies incentive compatibility and individual rationality.

1. Show that this allocation rule is monotone non-decreasing in value, if all of the bidders’ virtual value functions are non-decreasing (meaning their underlying value distributions are regular).

2. Use Myerson’s lemma to compute payments that guarantee incentive compatibility and individual rationality when using this allocation rule.

3. This allocation rule has an interesting property: a bidder with the highest bid/value may not win the good, even if their virtual value is positive. Construct an example where this happens. Your example should specify each player’s value distribution and realized value.

3 Optimal Revenue, and its dependency on n

In this problem, you will explore just how different total expected revenue can be between two auction formats:

1. The second-price, sealed-bid auction.

2. The revenue-maximizing second-price, sealed-bid auction.

In particular, you will implement these two auction formats, and then run a Monte Carlo simulation, where each bidder $i \in N$ draws values from a continuous uniform $U(0,1)$ distribution.

Answer the following questions, again assuming bidders values are drawn from a continuous uniform $U(0,1)$ distribution:

1. For $n = 2$, what is the expected revenue of the welfare-maximizing second-price, sealed-bid auction (with no reserve)?

2. For $n = 2$, what is the expected revenue of the revenue-maximizing second-price, sealed-bid auction (with a monopoly reserve)?

3. What is each bidder’s virtual value function, $\varphi_i$?

4. Is the virtual value function you computed strictly increasing?
5. What is the inverse of the virtual value function? (That is, what is \( v_i \) in terms of \( \varphi_i \)?)

6. In any programming language you like, implement a Monte Carlo simulation for the setup described, varying the number of bidders from 2 to 20. Report, for each \( n \in \{2, 3, \ldots, 19, 20\} \) the following:\(^1\)

(a) The number of samples you used.
(b) The average revenue from a second-price, sealed-bid auction.
(c) The average revenue from a revenue-maximizing second-price, sealed-bid auction.
(d) The difference in revenue between the revenue-maximizing auction, and the welfare-maximizing auction.

If implemented and run correctly, your simulations should generate expected revenues close to those of your answers to the corresponding analysis questions. Additionally, you should verify that revenue is non-decreasing as you increase the number of bidders. Run your simulations for long enough so that, if you were to plot your results, the curves would look relatively smooth.

For this part of the problem, we require that you submit your code. There will be instructions on how to do this announced on Piazza at least two days before the deadline. We will check your code to make sure that it produces the results that you report, and we will run MOSS to make sure that there are no violations of the collaboration policy. These are the only two ways in which we will use your code to determine your score for this problem.

4 Knapsack, Revisited

Recall the Knapsack Auction homework problem from Homework 3.

1. Prove that Alice’s improved allocation scheme is a \( 2 \)-approximation of the optimal welfare. \(^2\)

2. Restate Alice’s allocation scheme in terms of virtual values. This will give a \( 2 \)-approximation of the optimal expected revenue.

3. Argue that this allocation function is monotone non-decreasing in a bidder’s bid, assuming that the bidder’s virtual value function is non-decreasing. Use Myerson’s payment formula to produce DSIC payments. State the payments in terms of virtual values.\(^3\)
5 Sponsored Search

Recall the sponsored search auction setup: \( n \) bidders (online advertisers) are competing for one of \( k \) sponsored links on a page that results from a keyword search (e.g., “TV”). For each slot \( j \), there is an associated probability that a user will click on a link in that slot. This probability is called the **click-through-rate** (CTR).\(^3\) For slot \( j \), we denote the CTR by \( \alpha_j \), and we assume \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k \). If bidder \( i \) is allocated slot \( j \), we say that \( x_i = \alpha_j \).

Each bidder \( i \) has a valuation \( v_i \) that corresponds to how much they value a user clicking on their advertisement. Thus, if a user is allocated slot \( j \) and pays \( p_i \), their utility is given by

\[
\alpha_j v_i - p_i
\]  

The bidders’ valuations are distributed uniformly at random between 0 and 1.

Create an auction (i.e., allocation scheme and payment rule) that maximizes expected revenue and satisfies individual rationality, incentive compatibility, and ex-post feasibility. Prove that your auction has all of these properties. Because bidders have valuations that are independent of the CTRs of the slots, each bidder should submit just one bid, not one bid per slot. Only allocate one slot to each person.

\(^3\) In reality, the probability with which a link is clicked depends on both its slot and its contents. We assume that bidders take this into account in their valuations of each slot.