CSCI 1590
Intro to Computational Complexity
The Limited Power of Diagonalization

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Time Hierarchy Theorem

If the TM $M$ on input $x$ runs in time $t(x)$, the Universal TM $U$ defined previously can simulate this computation in time $O(t^2(x))$.

To see this, observe that $U$ may bounce back and forth between the description $\lfloor M \rfloor$ of $M$ at the beginning of a tape to the head position, taking at most $O(t(x))$ steps to simulate one step of $M$.

Theorem

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ and $g : \mathbb{N} \rightarrow \mathbb{N}$ be proper resource functions and let $f(n) \log f(n) = o(g(n))$. Then,

$$\text{DTIME}(f(n)) \subsetneq \text{DTIME}(g(n))$$
Time Hierarchy Theorem

Proof

The result uses the fact that a universal TM $U$ can simulate a TM $M$ on input of length $n$ in $O(n \log n)$ steps.

We prove weaker result, namely, that $\text{DTIME}(n) \subsetneq \text{DTIME}(n^{2.10})$.

Given an input string $x$, let $M_x$ be the TM with description $\lfloor M \rfloor = x$. If $x$ is not well-formed, let it represent the TM that has one state and accepts all inputs.

Let $D$ be a DTM that simulates $M_x$ with the universal TM $U$ on input $x$ for $|x|^{2.1}$ steps. If $M_x$ outputs an answer in $\{0, 1\}$ (accept, reject), let $D$ produce $D(x) = 1 - M_x(x)$. Otherwise, let $D(x) = 0$.

$D$ accepts a language $L \in \text{DTIME}(n^{2.10})$. We show that $L \not\in \text{DTIME}(n)$.
Proof (cont.)

Assume that there exists TM $T$ that decides $x \in L$ in time $cn$ for some constant $c > 0$, $n = |x|$. We show a contradiction.

Given $T$, for every $x \in \Sigma^*$, $T(x) = D(x)$. $U$ simulates $T$ on input $x$ in time at most $d|x|^2$ for some constant $d > 0$.

There exists $n'$ such that for $n \geq n'$, $n^{2.1} > dn^2$. Because $T$ is equivalent to an infinite set of TMs, there is a description $x$ of a TM equivalent to $T$ of length greater than $n'$. Given the definition of $D$, on input $x$, $D(x) = 1 - T(x) \neq T(x)$. We have a contradiction and conclude that $T$ does not exist.
Diagonalization uses two facts, a) each TM can be represented by a computable string and b) a universal TM exists that simulates another TM on its input with a small (logarithmic) overhead. These properties apply to oracle TMs. We show that oracle TMs can’t resolve whether or not $P = \text{NP}$.

**Definition**

An oracle Turing machine (OTM) $M$ is TM and an oracle $O \subseteq \{0, 1\}$. $M$ has three special states, $q_{oracle}$, $q_{yes}$, and $q_{no}$ and a special read/write tape. $M$ writes a string on this tape. When it enters state $q_{oracle}$, the oracle determines whether or not this string is in $O$. If so, it moves $M$ to state $q_{yes}$ in one step. Otherwise, it moves $M$ to $q_{no}$ in one step. $M$ can be deterministic or nondeterministic.
Definition

For \( O \subseteq \{0, 1\} \), \( P^O \) is the class of languages recognized by a PTIME DTM with oracle \( O \). Similarly, \( NP^O \) is the class recognized by a nondeterministic PTIME NTM with oracle \( O \).

Proposition

1. \( \text{coSAT} \) are “No” instances of \( \text{SAT} \). Then, \( \text{coSAT} \in P^{\text{SAT}} \).
2. If \( O \in P \), \( P^O = P \).
3. Let \( \text{EXPCOM} \) be the language described below.

\[ \{ < [M], x, 1^n > | M \text{ outputs 1 on } x \text{ in } 2^n \text{ steps} \} \]

Then, \( P^{\text{EXPCOM}} = NP^{\text{EXPCOM}} = \text{EXPTIME} \).
Oracle Turing Machines

Proof.

1. To decide the “No” instances of SAT, write an instance $\phi$ of SAT on the oracle tape. Flip the response of the oracle.

2. Clearly $\mathbf{P} \subseteq \mathbf{P}^O$. If $O \in \mathbf{P}$, the oracle is redundant; we can simply incorporate its TM into a TM in $\mathbf{P}$. Thus, $\mathbf{P}^O \subseteq \mathbf{P}$.

3. Clearly, $\mathbf{EXPTIME} \subseteq \mathbf{P}^{\mathsf{EXPCOM}}$ – the oracle permits an exponential-time computation in one step.

Let $M \in \mathbf{NP}^{\mathsf{EXPCOM}}$. In exponential time one can examine the exponentially many choices implied by a polynomial-length certificate and the polynomially many invocations of the $\mathsf{EXPCOM}$ oracle. Thus, $\mathbf{NP}^{\mathsf{EXPCOM}} \subseteq \mathbf{EXPTIME}$. It follows that

$$\mathbf{EXPTIME} \subseteq \mathbf{P}^{\mathsf{EXPCOM}} \subseteq \mathbf{NP}^{\mathsf{EXPCOM}} \subseteq \mathbf{EXPTIME}$$
Recall that diagonalization uses two facts, a) each TM can be represented by a computable string and b) a universal TM exists that simulates another TM on its input with a small (logarithmic) overhead. These properties apply to oracle TMs.

A universal oracle TM with oracle $O$, $OU$, exists that can simulate an arbitrary oracle TM using small (logarithmic) overhead using a computable description of an oracle TM.

**Theorem (Baker, Gill, Solovay 1975)**

There are oracles $O_1$ and $O_2$ such that $P^{O_1} = NP^{O_1}$ and $P^{O_2} \neq NP^{O_2}$.

Diagonalization alone does not suffice to separate $P$ from $NP$!
Proof

For the first statement, let $O_1 = \text{EXPCOM}$. For the second, we construct a language $B$. Let $U_B$ be the following unary language:

$$U_B = \{1^n \mid \text{some string of length } n \text{ is in } B\}$$

For every oracle $B$, $U_B \in \text{NP}^B$ because an NTM given $1^n$ can guess a string $x \in B$, $|x| = n$ and then use the oracle to verify it. We construct a $B$ such that $U_B \notin \text{P}^B$.

$B$ is constructed in stages. At $i$th stage, $1 \leq i$, strings are added based on oracle queries made by $i$th oracle TM $M_i^B$, $M_i$ with oracle $B$. The goal is to create $B$ such that $U_B$ cannot be decided in $\leq 2^n/5$ steps.
Under Relativization Both $\mathbf{P} = \mathbf{NP}$ and $\mathbf{P} \neq \mathbf{NP}$

Proof (cont.)

Initially $B$ is empty. At $i$th stage choose $n$ larger than the length of any string currently in $B$. Run $M^B_i$ on input $1^n$ for $2^n/5$ steps. If $M^B_i$ issues a query string whose status has been determined at earlier stage, give the same response to $M^B_i$.

If $M^B_i$ halts in $2^n/5$ steps on input $1^n$, we make sure that its answer is incorrect. Do this by not including any string of length $n$ in $B$ if $M^B_i$ accepts (ensures that $1^n$ is rejected) and by including some string of length $n$ in $B$ that has not been queried (ensures that $1^n$ is accepted) if it rejects. (Such a string exists since at most $2^n/5$ queries have been issued.)

Let $U_B$ be accepted by $M^B_i$. If it doesn’t halt in $2^n/5$ steps on input $1^n$, $U_B \not\in \mathbf{P}^B$. If $M^B_i$ halts in $2^n/5$ steps on input $1^n$, it also doesn’t accept $U_B$. It follows that $U_B$ is not in $\mathbf{P}^B$ or that $\mathbf{P}^B \neq \mathbf{NP}^B$. 

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