Problem 1

Consider the triangle of numbers:

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 3
 7 1
 2 4 6
 8 5 9 3
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We wish to find the maximum sum from the top to the bottom, moving down and to the left or right. For example, if you are at 4, you can move to 5 and 9 but nothing else.

a. What is the maximum sum in the triangle above?

b. Devise an $O(n)$ algorithm to find the maximum sum in an arbitrary triangle with $n$ numbers.

c. Prove the correctness of your algorithm.

d. Prove the run time of your algorithm.
Problem 2

A carpenter has a piece of wood of a certain length $L$ that must be cut at positions $\{a_1, a_2, ..., a_n\}$ where $a_i$ is the distance from the left end of the original piece of wood. Notice that after making the first cut, the carpenter now has two pieces of wood; after making the second cut, the carpenter has three pieces of wood, etc. Assume that the cost of making a cut in a piece of wood of length $\ell$ is equal to $\ell$, and is the same no matter which position in that piece of wood is being cut. The goal of the carpenter is to minimize his costs while performing all of the cuts.

a. Given a piece of wood of original length $L = 10$ and a set of positions to cut it at $\{a_1, a_2, ..., a_n\} = \{1, 4, 5, 8\}$, what is the sequence of cuts that minimizes the cost of all of the cuts?

b. Design a dynamic programming algorithm for the Carpentry problem which takes as input the original length $L$ and a set of cuts defined by $\{a_1, a_2, ..., a_n\}$ and outputs the minimal cost to make all of the cuts in $O(n^3)$ time.

c. Prove the correctness of your algorithm. As an indication, we expect that the proof will typically be about $\frac{1}{2}$ of a page long.

d. Prove that the running time of your algorithm is $O(n^3)$. As an indication, we expect that the proof will typically be only a few lines long.
Problem 3

The input to the BigWin problem is a list \((x_1, x_2, \ldots, x_n)\) of integers (positive, zero or negative). The output is the maximum value of \(x_i + x_{i+1} + \ldots + x_j\) for all possibilities for \(i\) and \(j\), \(1 \leq i \leq j \leq n\). For example, if \(n = 9\) and the input is \((3, -2, -5, 8, -4, 1, 6, -4, 1)\) then the output is \(x_4 + x_5 + x_6 + x_7 = 8 - 4 + 1 + 6 = 11\).

a. Give an \(O(n)\) algorithm for the following related BigMiddleWin problem: The input to this problem is a list \((x_1, x_2, \ldots, x_n)\) of integers (positive, zero or negative). The output is the maximum value of \(x_i + x_{i+1} + \ldots + x_j\) such that \([i, j]\) spans \(n/2\), that is, for all possibilities for \(i\) and \(j\) such that \(0 \leq i \leq n/2 \leq j \leq n\).

b. Design an algorithm for the BigWin problem with runtime \(O(n \log n)\).

c. Prove the correctness of your algorithms from parts a and b.

d. Prove the runtime of your algorithm from part b.

e. Now design an algorithm for the BigWin problem that has \(O(n)\) runtime.

f. Prove the correctness of your algorithms from part e.

g. Prove the runtime of your algorithm from part e.