Martingales
Hoeffding’s Bound

**Theorem**

Let $X_1, \ldots, X_n$ be independent random variables with $E[X_i] = \mu_i$ and $Pr(B_i \leq X_i \leq B_i + c_i) = 1$, then

$$Pr\left(\left|\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i\right| \geq \epsilon \right) \leq e^{-\frac{2\epsilon^2}{\sum_{i=1}^{n} c_i^2}}$$

Do we need independence?
Martingales

Definition

A sequence of random variables \( Z_0, Z_1, \ldots \) is a *martingale* with respect to the sequence \( X_0, X_1, \ldots \) if for all \( n \geq 0 \) the following hold:

1. \( Z_n \) is a function of \( X_0, X_1, \ldots, X_n \);
2. \( E[|Z_n|] < \infty \);
3. \( E[Z_{n+1}|X_0, X_1, \ldots, X_n] = Z_n \);

Definition

A sequence of random variables \( Z_0, Z_1, \ldots \) is a *martingale* when it is a martingale with respect to itself, that is

1. \( E[|Z_n|] < \infty \);
2. \( E[Z_{n+1}|Z_0, Z_1, \ldots, Z_n] = Z_n \);
Conditioning Defines a Probability Space

Let \((\Omega, Pr(\cdot))\) be a probability space.

Let \(B\) be an event in \(\Omega\), \(Pr(B) > 0\).

We show that \((B, Pr(\cdot | B))\) is a probability space.

1. For any \(E \subseteq B\),

\[
0 \leq Pr(E | B) = \frac{Pr(E \cap B)}{Pr(B)} \leq 1
\]

2. Let \(E_1\) and \(E_2\) be disjoint events in \(B\),

\[
Pr(E_1 \cup E_1 | B) = \frac{Pr((E_1 \cup E_2) \cap B)}{Pr(B)}
\]

\[
= \frac{Pr(E_1 \cap B)}{Pr(B)} + \frac{Pr(E_2 \cap B)}{Pr(B)}
\]

\[
= Pr(E_1 | B) + Pr(E_2 | B)
\]
## Conditional Expectation

### Definition

\[
E[Y \mid Z = z] = \sum_y y \Pr(Y = y \mid Z = z),
\]
where the summation is over all \( y \) in the range of \( Y \).

Note that \( E[Y \mid Z] \) is a random variable (a function of \( Z \)).

### Lemma

*For any random variables \( X \) and \( Y \),*

\[
E[X] = E_Y[E_X[X \mid Y]] = \sum_y \Pr(Y = y)E[X \mid Y = y],
\]

*where the sum is over all values in the range of \( Y \).*
Lemma

For any random variables $X$ and $Y$, 
\[
E[X] = E_Y[E_X[X | Y]] = \sum_y \Pr(Y = y)E[X | Y = y],
\]
where the sum is over all values in the range of $Y$.

Proof.

\[
\sum_y \Pr(Y = y)E[X | Y = y]
= \sum_y \Pr(Y = y) \sum_x x \Pr(X = x | Y = y)
= \sum_x \sum_y x \Pr(X = x | Y = y) \Pr(Y = y)
= \sum_x \sum_y x \Pr(X = x \cap Y = y) = \sum_x x \Pr(X = x) = E[X].
\]
Example

$Y$ - the number of students attending class, $Y \sim B(n, p)$

$X$ - the number of questions asked in class is $X|_{Y=y} \sim B(\lfloor \sqrt{y} \rfloor, q)$.

All events are independent

$E[X | Y = y] = q\lfloor \sqrt{y} \rfloor$ - a constant

$E[X | Y] = q\lfloor \sqrt{Y} \rfloor$ - a random variable

$$E[X] = E_Y[E_X[X | Y]] = E_Y[q\lfloor \sqrt{Y} \rfloor] = qE[\lfloor \sqrt{Y} \rfloor] \leq q\sqrt{E[Y]} = q\sqrt{np}$$
Martingales

Definition

A sequence of random variables $Z_0, Z_1, \ldots$ is a martingale with respect to the sequence $X_0, X_1, \ldots$ if for all $n \geq 0$ the following hold:

1. $Z_n$ is a function of $X_0, X_1, \ldots, X_n$;
2. $\mathbb{E}[|Z_n|] < \infty$;
3. $\mathbb{E}[Z_{n+1}|X_0, X_1, \ldots, X_n] = Z_n$;

Definition

A sequence of random variables $Z_0, Z_1, \ldots$ is a martingale when it is a martingale with respect to itself, that is

1. $\mathbb{E}[|Z_n|] < \infty$;
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Martingale Example

A series of fair games ($E[\text{gain}] = 0$), not necessarily independent..

Game 1: bet $1$.

Game $i > 1$: bet $2^i$ if won in round $i - 1$; bet $i$ otherwise.

$X_i =$ amount won in $i$th game. ($X_i < 0$ if $i$th game lost).

$Z_i =$ total winnings at end of $i$th game.
Example

\[ X_i = \text{amount won in } i\text{th game. (}X_i < 0 \text{ if } i\text{th game lost).} \]

\[ Z_i = \text{total winnings at end of } i\text{th game.} \]

\[ Z_1, Z_2, \ldots \text{ is martingale with respect to } X_1, X_2, \ldots \]

\[ E[X_i] = 0. \]

\[ E[Z_i] = \sum_{j=1}^{i} E[X_j] = 0 < \infty. \]

\[ E[Z_{i+1}|X_1, X_2, \ldots, X_i] = Z_i + E[X_{i+1}] = Z_i. \]
Efficient Market Hypothesis

The efficient markets hypothesis (EMH) maintains that market prices fully reflect all available information. Samuelson (1965), Fama (1963);

For simplicity assume an asset that is paying no dividend, and assume 0 interest rate (so value is not discounted in time).

Let \( X_t \) be the price of a unit asset at time \( t \).
If I know that at time \( t+1 \) the price will be \( X_{t+1} = c \), I will not sale the asset now for last than \( c \).
If I know that at time \( t+1 \) the price will be \( X_{t+1} = c \), I will not buy the asset now for more than \( c \).

\[
X_t = E[X_{t+1} \mid X_0, \ldots, X_t]
\]

\( X_0, X_1, \ldots, X_t \), is a martingale.
Gambling Strategies

I play series of fair games (win with probability $1/2$).

Game 1: bet $1$.

Game $i > 1$: bet $2^i$ if I won in round $i - 1$; bet $i$ otherwise.

$X_i =$ amount won in $i$th game. ($X_i < 0$ if $i$th game lost).

$Z_i =$ total winnings at end of $i$th game.

Assume that (before starting to play) I decide to quit after $k$ games: what are my expected winnings?
**Lemma**

Let $Z_0, Z_1, Z_2, \ldots$ be a martingale with respect to $X_0, X_1, \ldots$. For any fixed $n$,

$$E_{X[0:n]}[Z_n] = E_{X_0}[Z_0] .$$

$(X[0 : i] = X_0, \ldots, X_i)$

**Proof.**

Since $Z_i$ is a martingale $E_{X_i}[Z_i|X_0, X_1, \ldots, X_{i−1}] = Z_{i−1}$. Then

$$E_{X[0:i−1]}[Z_{i−1}] = E_{X[0:i−1]}[E_{X_i}[Z_i|X_0, X_1, \ldots, X_{i−1}]] = E_{X[0:i]}[Z_i]$$

Thus,

$$E_{X[0:n]}[Z_n] = E_{X[0:n−1]}[Z_{n−1}] = \ldots, = E[Z_0]$$
I play series of fair games (win with probability $1/2$).

Game 1: bet $1$.

Game $i > 1$: bet $2^i$ if I won in round $i - 1$; bet $i$ otherwise.

$X_i =$ amount won in $i$th game. ($X_i < 0$ if $i$th game lost).

$Z_i =$ total winnings at end of $i$th game.

Assume that (before starting to gamble) we decide to quit after $k$ games: what are my expected winnings?

$E[Z_k] = E[Z_1] = 0.$
Same gambling game. What happens if I:

- play a random number of games?
- decide to stop only when I have won $1000?
Definition

A non-negative, integer random variable $T$ is a stopping time for the sequence $Z_0, Z_1, \ldots$ if the event “$T = n$” depends only on the value of random variables $Z_0, Z_1, \ldots, Z_n$.

Intuition: corresponds to a strategy for determining when to stop a sequence based only on values seen so far.

In the gambling game:

- *first time I win 10 games in a row*: is a stopping time;
- *the last time when I win*: is not a stopping time.
Consider again the gambling game: let $T$ be a stopping time.

$Z_i =$ total winnings at end of $i$th game.

What are my winnings at the stopping time, i.e. $E[Z_T]$?

Fair game: $E[Z_k] = E[Z_0] = 0$?

"$T =$first time my total winnings are at least $\$1000$" is a stopping time, and $E[Z_T] > 1000$...
Martingale Stopping Theorem

Theorem

If $Z_0, Z_1, \ldots$ is a martingale with respect to $X_1, X_2, \ldots$ and if $T$ is a stopping time for $X_1, X_2, \ldots$ then (if $T$ is finite),

$$E[Z_T] = E[Z_0]$$

whenever one of the following holds:

1. there is a constant $c$ such that, for all $i$, $|Z_i| \leq c$;
2. $T$ is bounded;
3. $E[T] < \infty$, and there is a constant $c$ such that $E[|Z_{i+1} - Z_i| |X_1, \ldots, X_i] < c$. 
Proof of Martingale Stopping Theorem (Sketch)

Define a sequence $Y_0, Y_1, \ldots$ such that

$$Y_i = \begin{cases} Z_i & \text{if } T > i \\ Z_T & \text{if } T \leq i \end{cases}$$

Lemma

The sequence $Y_0, Y_1, \ldots$ is a martingale with respect to $Z_0, Z_1, \ldots$.

Proof.

1. $Y_n$ is determined by $Z_0, \ldots, Z_n$.
2. $E[|Y_n|] \leq \max_{0 \leq i \leq n} E[|X_i|] \leq \sum_{i=1}^n E[|X_i|] < \infty$
3. $E[Y_{n+1}|Z_0, Z_1, \ldots, Z_n] = Y_n + E[Z_{n+1}[(Y_{n+1} - Y_n)Pr(T > n)]] = Y_n + E[(Z_{n+1} - Z_n)]Pr(T > n) = Y_n;

Since $Pr(T > n)$ is independent of $Z_{n+1}$, and $E[(Z_{n+1} - Z_n)] = 0$. 

\qed
Since $Y_0, Y_1, \ldots$ is a martingale, for any $n \geq 0$, $E[Y_n] = E[Z_0]$, and

$$\lim_{n \to \infty} E[Y_n] = E[Y_0] = E[Z_0].$$

We want to show that $E[Z_T] = \lim_{n \to \infty} E[Y_n] = E[Z_0]$.

We use a simple version of the Uniform Convergence Theorem:

**Theorem**

Let $W_0, W_1, \ldots$ be a sequence of random variables such that $\lim_{n \to \infty} W_n = W$ (pointwise), and $\max_i |W_i| \leq M$, where $M$ is either a constant or a random variable with $E[|M|] < \infty$, then

$$\lim_{n \to \infty} E[W_n] = E[W].$$
Proof of Martingale Stopping Theorem (Sketch)

Since \( T \) is finite, \( \lim_{n \to \infty} Y_n = \lim_{n \to \infty} Z_{\text{min}(n, T)} = Z_T \).

We need to show that \( |Y_n| \leq M \).

1. There is a constant \( c \) such that, for all \( i \), \( |Z_i| \leq c \) - 
   \[ |Y_n| \leq \max_{0 \leq i \leq n} |Z_i| \leq c, \quad c = M < \infty. \]

2. \( T \) is bounded - 
   \[ |Y_n| \leq \max_{0 \leq i \leq \max T} |Z_i| \leq M < \infty \]

3. \( \mathbb{E}[T] < \infty \), and there is a constant \( c \) such that
   \[ \mathbb{E}[|Z_{i+1} - Z_i| \mid X_1, \ldots, X_i] < c \]

\[
Y_n = Z_0 + \sum_{i=1}^{\infty} (Z_{i+1} - Z_i)1_{i \leq T} \leq |Z_0| + \sum_{i=1}^{\infty} |Z_{i+1} - Z_i|1_{i \leq T} = M.
\]

\[
\mathbb{E}[|M|] \leq \mathbb{E}[|Z_0|] + \sum_{i=1}^{\infty} \mathbb{E}[\mathbb{E}[|Z_{i+1} - Z_i| \mid X_1, \ldots, X_i]1_{i \leq T}]
\]

\[
\leq \mathbb{E}[|Z_0|] + c \sum_{i=1}^{\infty} \mathbb{P}(T \geq i)
\]

\[
\leq \mathbb{E}[|Z_0|] + c\mathbb{E}[T] < \infty
\]
We play a sequence of fair game with the following stopping rules:

2. $T$ is the first time we made $1000$: $E[T]$ is unbounded.
3. We double until the first win. $E[T] = 2$ but $E[|Z_{i+1} - Z_i| |X_1, \ldots, X_i]$ is unbounded.
Example: The Gambler’s Ruin

- Consider a sequence of independent, fair 2-player gambling games.
- In each round, each player wins or loses $1 with probability \( \frac{1}{2} \).
- \( X_i \) = amount won by player 1 on \( i \)th round.
  - If player 1 has lost in round \( i \): \( X_i < 0 \).
- \( Z_i \) = total amount won by player 1 after \( i \)th rounds.
  - \( Z_0 = 0 \).
- Game ends when one player runs out of money
  - Player 1 must stop when she loses net \( \ell_1 \) dollars (\( Z_t = -\ell_1 \))
  - Player 2 terminates when she loses net \( \ell_2 \) dollars (\( Z_t = \ell_2 \)).
- \( q \) = probability game ends with player 1 winning \( \ell_2 \) dollars.
Example: The Gambler’s Ruin

- \( T \) = first time player 1 wins \( \ell_2 \) dollars or loses \( \ell_1 \) dollars.
  - \( T \) is a stopping time for \( X_1, X_2, \ldots \).
- \( Z_0, Z_1, \ldots \) is a martingale.
  - \( Z_i \)'s are bounded.
- Martingale Stopping Theorem: \( \mathbb{E}[Z_T] = \mathbb{E}[Z_0] = 0 \).

\[
\mathbb{E}[Z_T] = q\ell_2 - (1 - q)\ell_1 = 0
\]
\[
q = \frac{\ell_1}{\ell_1 + \ell_2}
\]
Example: A Ballot Theorem

- Candidate A and candidate B run for an election.
  - Candidate A gets \( a \) votes.
  - Candidate B gets \( b \) votes.
  - \( a > b \).

- Votes are counted in *random order*:
  - chosen from all permutations on all \( n = a + b \) votes.

- What is the probability that A is always ahead in the count?
Example: A Ballot Theorem

- $S_i =$ number of votes A is leading by after $i$ votes counted
  - If A is trailing: $S_i < 0$.
  - $S_n = a - b$.
  - For $0 \leq k \leq n - 1$: $X_k = \frac{S_{n-k}}{n-k}$.
  - Consider $X_0, X_1, \ldots, X_n$.
    - This sequence goes backward in time!

$$E[X_k|X_0, X_1, \ldots, X_{k-1}] = ?$$
Example: A Ballot Theorem

\[ E[X_k | X_0, X_1, \ldots, X_{k-1}] = ? \]

- Conditioning on \( X_0, X_1, \ldots, X_{k-1} \): equivalent to conditioning on \( S_n, S_{n-1}, \ldots, S_{n-k+1} \).
- \( a_i \) = number of votes for A after first \( i \) votes are counted.
- \((n - k + 1)\)th vote: random vote among these first \( n - k + 1 \) votes.

\[
S_{n-k} = \begin{cases} 
S_{n-k+1} + 1 & \text{if } (n - k + 1)\text{th vote is for B} \\
S_{n-k+1} - 1 & \text{if } (n - k + 1)\text{th vote is for A}
\end{cases}
\]

\[
S_{n-k} = \begin{cases} 
S_{n-k+1} + 1 & \text{with prob. } \frac{n-k+1-a_{n-k+1}}{n-k+1} \\
S_{n-k+1} - 1 & \text{with prob. } \frac{a_{n-k+1}}{n-k+1}
\end{cases}
\]
\[
E[S_{n-k} | S_{n-k+1}] = (S_{n-k+1} + 1) \frac{n - k + 1 - a_{n-k+1}}{(n - k + 1)} \\
+ (S_{n-k+1} - 1) \frac{a_{n-k+1}}{(n - k + 1)} \\
= S_{n-k+1} \frac{n - k}{n - k + 1}
\]

(Since \(2a_{n-k+1} - n - k + 1 = S_{n-k+1}\))

\[
E[X_k | X_0, X_1, \ldots, X_{k-1}] = E \left[ \frac{S_{n-k}}{n - k} \middle| S_n, \ldots, S_{n-k+1} \right] \\
= \frac{S_{n-k+1}}{n - k + 1} \\
= X_{k-1}
\]

\(\implies \) \(X_0, X_1, \ldots, X_n\) is a martingale.
Example: A Ballot Theorem

\[ T = \begin{cases} \min \{ k : X_k = 0 \} & \text{if such } k \text{ exists} \\ n - 1 & \text{otherwise} \end{cases} \]

- \( T \) is a stopping time.
- \( T \) is bounded.
- Martingale Stopping Theorem:

\[ E[X_T] = E[X_0] = \frac{E[S_n]}{n} = \frac{a - b}{a + b} . \]

Two cases:

1. \( A \) leads throughout the count.
2. \( A \) does not lead throughout the count.
1. A leads throughout the count.

For $0 \leq k \leq n - 1$: $S_{n-k} > 0$, then $X_k > 0$.

$T = n - 1$.

$X_T = X_{n-1} = S_1$.

A gets the first vote in the count: $S_1 = 1$, then $X_T = 1$.

2. A does not lead throughout the count.

For some $k$: $S_k = 0$. Then $X_k = 0$.

$T = k < n - 1$.

$X_T = 0$. 
Example: A Ballot Theorem

Putting all together:

1. \( A \) leads throughout the count: \( X_T = 1 \).
2. \( A \) does not lead throughout the count: \( X_T = 0 \)

\[
E[X_T] = \frac{a - b}{a + b} = 1 \cdot \Pr(\text{Case 1}) + 0 \cdot \Pr(\text{Case 2}) .
\]

That is

\[
\Pr(\text{A leads throughout the count}) = \frac{a - b}{a + b} .
\]