CS155/254: Probabilistic Methods in Computer Science

Class 7: Packet routing on an hypercube network
Online Class

• Attend synchronous class (if possible), participate, ask questions

On Zoom:

• Use the largest display you have. Your phone is not a good choice
• Join Zoom with video ON and audio Muted
• View options: Fit to Window
• To participate:
  • Use Raise Hand in Reactions
  • Unmute and talk
  • Chat to Everyone or Host.
Packet Routing on Parallel Computer

Communication network:
Packet Routing on Parallel Computer

Communication network:

- nodes - processors, switching nodes;
- edges - communication links.
Model and Computational problem

- An edge \((v, w)\) corresponds to two directed edges, \(v \rightarrow w\) and \(w \rightarrow v\).
- Up to one packet can cross an edge per step, each packet can cross up to one edge per step.
- A permutation communication request: each node is the source and destination of exactly one packet.
- What is the time to route an arbitrary permutation on an \(N\) node network?
The $n$-cube

The 3-cube:

The 4-cube:
The $n$-cube

The $n$-cube:
$N = 2^n$ nodes: $0, 1, 2, \ldots, 2^n - 1$.

Let $\bar{x} = (x_1, \ldots, x_n)$ be the number of node $x$ in binary.

Nodes $x$ and $y$ are connected by an edge iff their binary representations differ in exactly one bit.

Bit-wise routing: correct bit $i$ in the $i$-th transition - route has length $\leq n$.

**Problem:** Assume that a packet from $(x_1, \ldots, x_{n/2}, 0, 0, \ldots, 0)$ is routed to $(0, 0, \ldots, 0, x_1, \ldots, x_{n/2})$, for all possible assignments of $x_1, \ldots, x_{n/2}$.

We have $2^{n/2} = \sqrt{N}$ packets traversing node $(0, \ldots, 0)$.
There is an edge that is traversed by $\sqrt{N}/n$ packets.
Randomized Packet Routing Algorithm on the $n$-cube

Two phase routing algorithm:

1. Send packet to a randomly chosen destination.
2. Send packet from randomly chosen destination to real destination.

Path: Correct the bits, $x_1$ to $x_n$.
Queue policy: Any greedy queuing method - if a queue to an edge is not empty one packet traverse the edge the edge.

**Theorem**

The two phase routing algorithm routes an arbitrary permutation on the $n$-cube in $O(\log N) = O(n)$ parallel steps with high probability.
Theorem

The two phase routing algorithm routes an arbitrary permutation on the $n$-cube in $O(\log N) = O(n)$ parallel steps with high probability.

- We focus first on phase 1. We bound the routing time of a given packet $M$.
- Let $e_1, \ldots, e_m$ be the $m \leq n$ edges traversed by a given packet $M$ in phase 1.
- Let $X(e)$ be the total number of packets that traverse edge $e$ at that phase.
- Let $T(M)$ be the number of steps till $M$ finished phase 1.
Lemma

\[ T(M) \leq \sum_{i=1}^{m} X(e_i). \]

- We call any path \( P = (e_1, e_2, \ldots, e_m) \) of \( m \leq n \) edges that follows the bit fixing algorithm a possible packet path.
- We denote the corresponding nodes \( v_0, v_1, \ldots, v_m \), with \( e_i = (v_{i-1}, v_i) \).
- For any possible packet path \( P \), let \( T(P) = \sum_{i=1}^{m} X(e_i) \).
• If phase I takes more than $T$ steps then for some possible packet path $P$,

\[ T(P) \geq T \]

• There are at most $2^n \cdot 2^n = 2^{2n}$ possible packet paths.
• Assume that $e_k$ connects $(a_1, ..., a_i, ..., a_n)$ to $(a_1, .., \bar{a}_i, ..., a_n)$.
• Only packets that started in address

\[ (\ast, ..., \ast, a_i, ..., a_n) \]

can traverse edge $e_k$, and only if their destination addresses are

\[ (a_1, ...., a_{i-1}, \bar{a}_i, \ast, ...., \ast) \]

• There are no more than $2^{i-1}$ possible packets, each has probability $2^{-i}$ to traverse $e_i$. 
• There are no more than $2^{i-1}$ possible packets, each has probability $2^{-i}$ to traverse $e_i$.

\[
E[X(e_i)] \leq 2^{i-1} \cdot 2^{-i} = \frac{1}{2}.
\]

• 

\[
E[T(P)] \leq \sum_{i=1}^{m} E[X(e_i)] \leq \frac{1}{2} \cdot m \leq n.
\]

• **Problem:** The $X(e_i)$’s are not independent.
• A packet is active with respect to possible packet path $P$ if it ever use an edge of $P$.

• For $k = 1, \ldots, N$, let $H_k = 1$ if the packet starting at node $k$ is active, and $H_k = 0$ otherwise.

• The $H_k$ are independent, since each $H_k$ depends only on the choice of the intermediate destination of the packet starting at node $k$, and these choices are independent for all packets.

• Let $H = \sum_{k=1}^{N} H_k$ be the total number of active packets.

• $E[H] \leq E[T(P)] \leq n$

• Since $H$ is the sum of independent 0–1 random variables we can apply the Chernoff bound

$$\Pr(H \geq 6n) \leq \Pr(H \geq 6E[H]) \leq 2^{-6n}.$$
For a given possible packet path $P$,

\[
\Pr(T(P) \geq 30n) \leq \Pr(T(P) \geq 30n \mid H \geq 6n) \Pr(H \geq 6n) \\
+ \Pr(T(P) \geq 30n \mid H < 6n) \Pr(H < 6n) \\
\leq \Pr(H \geq 6n) + \Pr(T(P) \geq 30n \mid H < 6n) \\
\leq 2^{-6n} + \Pr(T(P) \geq 30n \mid H < 6n).
\]

We use:

\[
\Pr(A) = \Pr(A \mid B) \Pr(B) \\
+ \Pr(A \mid \bar{B}) \Pr(\bar{B}) \\
\leq \Pr(B) + \Pr(A \mid \bar{B})
\]
Lemma

If a packet leaves a path (of another packet) it cannot return to that path in the same phase.

Proof.

Leaving a path at the $i$-th transition implies different $i$-th bit, this bit cannot be changed again in that phase.

Lemma

The number of transitions that a packet takes on a given path is distributed $G\left(\frac{1}{2}\right)$.

Proof.

The packet has probability $1/2$ of leaving the path in each transition.
The probability that the active packets cross edges of $P$ more than $30n$ times is less than the probability that a fair coin flipped $36n$ times comes up heads less than $6n$ times. Letting $Z$ be the number of heads in $36n$ fair coin flips, we now apply the Chernoff bound:

\[
\Pr(T(P) \geq 30n \mid H \leq 6n) \leq \Pr(Z \leq 6n) \\
\leq e^{-18n(2/3)^2/2} = e^{-4n} \leq 2^{-3n-1}.
\]

\[
\Pr(T(P) \geq 30n) \leq \Pr(H \geq 6n) + \Pr(T(P) \geq 30n \mid H \leq 6n) \\
\leq 2^{-6n} + 2^{-3n-1} \leq 2^{-3n}
\]
As there are at most $2^{2n}$ possible packet paths in the hypercube, the probability that there is any possible packet path for which $T(P) \geq 30n$ is bounded by

$$2^{2n}2^{-3n} = 2^{-n} = O(N^{-1}).$$
The proof of phase 2 is by symmetry:

The proof of phase 1 argued about the number of packets crossing a given path, no “timing” considerations.

The path from “one packet per node” to random locations is similar to random locations to “one packet per node” in reverse order.

Thus, the distribution of the number of packets that crosses a path of a given packet is the same.
Oblivious Routing

**Definition**
A routing algorithm is **oblivious** if the path taken by one packet is independent of the source and destinations of any other packets in the system.

**Theorem**
*Given an* $N$*-node network with maximum degree* $d$*- the routing time of any deterministic oblivious routing scheme is*

$$\Omega \left( \sqrt{\frac{N}{d^3}} \right).$$