

Large Deviation Bounds

A typical probability theory statement:

Theorem (The Central Limit Theorem)

Let X_1, \dots, X_n be independent identically distributed random variables with common mean μ and variance σ^2 . Then

$$\lim_{n \rightarrow \infty} \Pr\left(\frac{\frac{1}{n} \sum_{i=1}^n X_i - \mu}{\sigma/\sqrt{n}} \leq z\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-t^2/2} dt.$$

A typical CS probabilistic tool:

Theorem (Chernoff Bound)

Let X_1, \dots, X_n be independent Bernoulli random variables such that $\Pr(X_i = 1) = p_i$. Let $\mu = \frac{1}{n} \sum_{i=1}^n p_i$, then

$$\Pr\left(\frac{1}{n} \sum_{i=1}^n X_i \geq (1 + \delta)\mu\right) \leq e^{-\mu n \delta^2 / 3}.$$

Chernoff's vs. Chebyshev's Inequality

Assume for all i we have $p_i = p; 1 - p_i = q$.

$$\mu = \mathbf{E}[X] = np$$

$$\text{Var}[X] = npq$$

If we use Chebyshev's Inequality we get

$$\Pr(|X - \mu| > \delta\mu) \leq \frac{npq}{\delta^2\mu^2} = \frac{npq}{\delta^2 n^2 p^2} = \frac{q}{\delta^2\mu}$$

Chernoff bound gives

$$\Pr(|X - \mu| > \delta\mu) \leq 2e^{-\mu\delta^2/3}.$$

The Basic Idea of Large Deviation Bounds:

For any random variable X , by Markov inequality we have:

For any $t > 0$,

$$\Pr(X \geq a) = \Pr(e^{tX} \geq e^{ta}) \leq \frac{\mathbf{E}[e^{tX}]}{e^{ta}}.$$

Similarly, for any $t < 0$

$$\Pr(X \leq a) = \Pr(e^{tX} \geq e^{ta}) \leq \frac{\mathbf{E}[e^{tX}]}{e^{ta}}.$$

Theorem (Markov Inequality)

If a random variable X is non-negative ($X \geq 0$) then

$$\text{Prob}(X \geq a) \leq \frac{E[X]}{a}.$$

The General Scheme:

We obtain specific bounds for particular conditions/distributions by

- 1 computing $E[e^{tX}]$
- 2 optimize

$$Pr(X \geq a) \leq \min_{t>0} \frac{E[e^{tX}]}{e^{ta}}$$

$$Pr(X \leq a) \leq \min_{t<0} \frac{E[e^{tX}]}{e^{ta}}.$$

- 3 simplify

Chernof Bound - Large Deviation Bound

Theorem

Let X_1, \dots, X_n be independent, identically distributed, 0 – 1 random variables with $Pr(X_i = 1) = E[X_i] = p$. Let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$, then for any $\delta \in [0, 1]$ we have

$$Prob(\bar{X}_n \geq (1 + \delta)p) \leq e^{-np\delta^2/3}$$

and

$$Prob(\bar{X}_n \leq (1 - \delta)p) \leq e^{-np\delta^2/2}.$$

Chernof Bound - Large Deviation Bound

Theorem

Let X_1, \dots, X_n be independent, 0 – 1 random variables with $Pr(X_i = 1) = E[X_i] = p_i$. Let $\mu = \sum_{i=1}^n p_i$, then for any $\delta \in [0, 1]$ we have

$$Prob\left(\sum_{i=1}^n X_i \geq (1 + \delta)\mu\right) \leq e^{-\mu\delta^2/3}$$

and

$$Prob\left(\sum_{i=1}^n X_i \leq (1 - \delta)\mu\right) \leq e^{-\mu\delta^2/2}.$$

Consider n coin flips. Let X be the number of heads.
Markov Inequality gives

$$\Pr\left(X \geq \frac{3n}{4}\right) \leq \frac{n/2}{3n/4} \leq \frac{2}{3}.$$

Using the Chebyshev's bound we have:

$$\Pr\left(\left|X - \frac{n}{2}\right| \geq \frac{n}{4}\right) \leq \frac{4}{n}.$$

Using the Chernoff bound in this case, we obtain

$$\begin{aligned} \Pr\left(\left|X - \frac{n}{2}\right| \geq \frac{n}{4}\right) &= \Pr\left(X \geq \frac{n}{2}\left(1 + \frac{1}{2}\right)\right) \\ &\quad + \Pr\left(X \leq \frac{n}{2}\left(1 - \frac{1}{2}\right)\right) \\ &\leq e^{-\frac{1}{3}\frac{n}{2}\frac{1}{4}} + e^{-\frac{1}{2}\frac{n}{2}\frac{1}{4}} \leq 2e^{-\frac{n}{24}}. \end{aligned}$$

Moment Generating Function

Definition

The moment generating function of a random variable X is defined for any real value t as

$$M_X(t) = \mathbf{E}[e^{tX}].$$

Theorem

Let X be a random variable with moment generating function $M_X(t)$. Assuming that exchanging the expectation and differentiation operands is legitimate, then for all $n \geq 1$

$$\mathbf{E}[X^n] = M_X^{(n)}(0),$$

where $M_X^{(n)}(0)$ is the n -th derivative of $M_X(t)$ evaluated at $t = 0$.

Proof.

$$M_X^{(n)}(t) = \mathbf{E}[X^n e^{tX}].$$

Computed at $t = 0$ we get

$$M_X^{(n)}(0) = \mathbf{E}[X^n].$$



Theorem

Let X and Y be two random variables. If

$$M_X(t) = M_Y(t)$$

for all $t \in (-\delta, \delta)$ for some $\delta > 0$, then X and Y have the same distribution.

Theorem

If X and Y are independent random variables then

$$M_{X+Y}(t) = M_X(t)M_Y(t).$$

Proof.

$$M_{X+Y}(t) = \mathbf{E}[e^{t(X+Y)}] = \mathbf{E}[e^{tX}] \mathbf{E}[e^{tY}] = M_X(t)M_Y(t).$$



Chernoff Bound for Sum of Bernoulli Trials

Theorem

Let X_1, \dots, X_n be independent Bernoulli random variables such that $\Pr(X_i = 1) = p_i$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \sum_{i=1}^n p_i$.

- For any $\delta > 0$,

$$\Pr(X \geq (1 + \delta)\mu) \leq \left(\frac{e^\delta}{(1 + \delta)^{1+\delta}} \right)^\mu. \quad (1)$$

- For $0 < \delta \leq 1$,

$$\Pr(X \geq (1 + \delta)\mu) \leq e^{-\mu\delta^2/3}. \quad (2)$$

- For $R \geq 6\mu$,

$$\Pr(X \geq R) \leq 2^{-R}. \quad (3)$$

Chernoff Bound for Sum of Bernoulli Trials

Let X_1, \dots, X_n be a sequence of independent Bernoulli trials with $\Pr(X_i = 1) = p_i$. Let $X = \sum_{i=1}^n X_i$, and let

$$\mu = \mathbf{E}[X] = \mathbf{E}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbf{E}[X_i] = \sum_{i=1}^n p_i.$$

For each X_i :

$$\begin{aligned} M_{X_i}(t) &= \mathbf{E}[e^{tX_i}] \\ &= p_i e^t + (1 - p_i) \\ &= 1 + p_i(e^t - 1) \\ &\leq e^{p_i(e^t - 1)}. \end{aligned}$$

$$M_{X_i}(t) = \mathbf{E}[e^{tX_i}] \leq e^{p_i(e^t-1)}.$$

Taking the product of the n generating functions we get for $X = \sum_{i=1}^n X_i$

$$\begin{aligned} M_X(t) &= \prod_{i=1}^n M_{X_i}(t) \\ &\leq \prod_{i=1}^n e^{p_i(e^t-1)} \\ &= e^{\sum_{i=1}^n p_i(e^t-1)} \\ &= e^{(e^t-1)\mu} \end{aligned}$$

$$M_X(t) = \mathbf{E}[e^{tX}] = e^{(e^t-1)\mu}$$

Applying Markov's inequality we have for any $t > 0$

$$\begin{aligned} Pr(X \geq (1 + \delta)\mu) &= Pr(e^{tX} \geq e^{t(1+\delta)\mu}) \\ &\leq \frac{\mathbf{E}[e^{tX}]}{e^{t(1+\delta)\mu}} \\ &\leq \frac{e^{(e^t-1)\mu}}{e^{t(1+\delta)\mu}} \end{aligned}$$

For any $\delta > 0$, we can set $t = \ln(1 + \delta) > 0$ to get:

$$Pr(X \geq (1 + \delta)\mu) \leq \left(\frac{e^\delta}{(1 + \delta)^{(1+\delta)}} \right)^\mu .$$

This proves (1).

We show that for $0 < \delta < 1$,

$$\frac{e^\delta}{(1+\delta)^{(1+\delta)}} \leq e^{-\delta^2/3}$$

or that $f(\delta) = \delta - (1+\delta)\ln(1+\delta) + \delta^2/3 \leq 0$
in that interval. Computing the derivatives of $f(\delta)$ we get

$$f'(\delta) = 1 - \frac{1+\delta}{1+\delta} - \ln(1+\delta) + \frac{2}{3}\delta = -\ln(1+\delta) + \frac{2}{3}\delta,$$

$$f''(\delta) = -\frac{1}{1+\delta} + \frac{2}{3}.$$

$f''(\delta) < 0$ for $0 \leq \delta < 1/2$, and $f''(\delta) > 0$ for $\delta > 1/2$.

$f'(\delta)$ first decreases and then increases over the interval $[0, 1]$.

Since $f'(0) = 0$ and $f'(1) < 0$, $f'(\delta) \leq 0$ in the interval $[0, 1]$.

Since $f(0) = 0$, we have that $f(\delta) \leq 0$ in that interval.

This proves (2).

For $R \geq 6\mu$, $\delta \geq 5$.

$$\begin{aligned} \Pr(X \geq (1 + \delta)\mu) &\leq \left(\frac{e^\delta}{(1 + \delta)^{(1+\delta)}} \right)^\mu \\ &\leq \left(\frac{e}{6} \right)^R \\ &\leq 2^{-R}, \end{aligned}$$

that proves (3).

Theorem

Let X_1, \dots, X_n be independent Bernoulli random variables such that $\Pr(X_i = 1) = p_i$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbf{E}[X]$.

For $0 < \delta < 1$:

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$$\Pr(X \leq (1 - \delta)\mu) \leq \left(\frac{e^{-\delta}}{(1 - \delta)(1 - \delta)} \right)^\mu. \quad (4)$$

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$$\Pr(X \leq (1 - \delta)\mu) \leq e^{-\mu\delta^2/2}. \quad (5)$$

Using Markov's inequality, for any $t < 0$,

$$\begin{aligned} \Pr(X \leq (1 - \delta)\mu) &= \Pr(e^{tX} \geq e^{(1-\delta)t\mu}) \\ &\leq \frac{\mathbf{E}[e^{tX}]}{e^{t(1-\delta)\mu}} \\ &\leq \frac{e^{(e^t-1)\mu}}{e^{t(1-\delta)\mu}} \end{aligned}$$

For $0 < \delta < 1$, we set $t = \ln(1 - \delta) < 0$ to get:

$$\Pr(X \leq (1 - \delta)\mu) \leq \left(\frac{e^{-\delta}}{(1 - \delta)^{(1-\delta)}} \right)^\mu$$

This proves (4).

We need to show:

$$f(\delta) = -\delta - (1 - \delta) \ln(1 - \delta) + \frac{1}{2}\delta^2 \leq 0.$$

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Differentiating $f(\delta)$ we get

$$\begin{aligned} f'(\delta) &= \ln(1 - \delta) + \delta, \\ f''(\delta) &= -\frac{1}{1 - \delta} + 1. \end{aligned}$$

Since $f''(\delta) < 0$ for $\delta \in (0, 1)$, $f'(\delta)$ decreasing in that interval. Since $f'(0) = 0$, $f'(\delta) \leq 0$ for $\delta \in (0, 1)$. Therefore $f(\delta)$ is non increasing in that interval.

$f(0) = 0$. Since $f(\delta)$ is non increasing for $\delta \in [0, 1)$, $f(\delta) \leq 0$ in that interval, and (5) follows.

Example: Coin Flips

Let X be the number of heads in a sequence of n independent fair coin flips.

Markov Inequality gives

$$\Pr\left(X \geq \frac{3n}{4}\right) \leq \frac{n/2}{3n/4} \leq \frac{2}{3}.$$

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Using the Chernoff bound in this case, we obtain

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Example: Coin flips

Theorem (The Central Limit Theorem)

Let X_1, \dots, X_n be independent identically distributed random variables with common mean μ and variance σ^2 . Then

$$\lim_{n \rightarrow \infty} \Pr\left(\frac{\frac{1}{n} \sum_{i=1}^n X_i - \mu}{\sigma/\sqrt{n}} \leq z\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-t^2/2} dt.$$

$\Phi(2.23) = 0.99$, thus, $\lim_{n \rightarrow \infty} \Pr\left(\frac{\frac{1}{n} \sum_{i=1}^n X_i - \mu}{\sigma/\sqrt{n}} \leq 2.23\right) = 0.99$

For coin flips:

$$\lim_{n \rightarrow \infty} \Pr\left(\frac{\frac{1}{n} \sum_{i=1}^n X_i - 1/2}{1/(2\sqrt{n})} \leq 2.23\right) = 0.99$$

$$\lim_{n \rightarrow \infty} \Pr\left(\frac{\sum_{i=1}^n X_i - n/2}{\sqrt{n}/2} \geq 2.23\right) = 0.01$$

$$\lim_{n \rightarrow \infty} \Pr\left(\sum_{i=1}^n X_i - \frac{n}{2} \geq 2.23\sqrt{n}/2\right) = 0.01$$

$$\Phi(3.5) \approx 0.999, \lim_{n \rightarrow \infty} \Pr\left(\sum_{i=1}^n X_i - \frac{n}{2} \geq 3.5\sqrt{n}/2\right) = 0.001$$

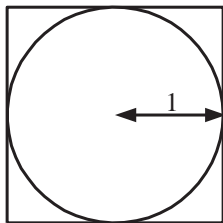
Example: Coin flips

Let X be the number of heads in a sequence of n independent fair coin flips.

$$\begin{aligned} & Pr \left(\left| X - \frac{n}{2} \right| \geq \frac{1}{2} \sqrt{6n \ln n} \right) \\ &= Pr \left(X \geq \frac{n}{2} \left(1 + \sqrt{\frac{6 \ln n}{n}} \right) \right) \\ &+ Pr \left(X \leq \frac{n}{2} \left(1 - \sqrt{\frac{6 \ln n}{n}} \right) \right) \\ &\leq e^{-\frac{1}{3} \frac{n}{2} \frac{6 \ln n}{n}} + e^{-\frac{1}{2} \frac{n}{2} \frac{6 \ln n}{n}} \leq \frac{2}{n}. \end{aligned}$$

Note that the standard deviation is $\sqrt{n/4}$

Example: estimate the value of π



- Choose X and Y independently and uniformly at random in $[0, 1]$.
- Let

$$Z = \begin{cases} 1 & \text{if } \sqrt{X^2 + Y^2} \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

- $\frac{1}{2} \leq p = \Pr(Z = 1) = \frac{\pi}{4} \leq 1$.
- $4\mathbf{E}[Z] = \pi$.

- Let Z_1, \dots, Z_m be the values of m independent experiments.
 $W_m = \sum_{i=1}^m Z_i$.

-

$$\mathbf{E}[W_m] = \mathbf{E} \left[\sum_{i=1}^m Z_i \right] = \sum_{i=1}^m \mathbf{E}[Z_i] = \frac{m\pi}{4},$$

- $W'_m = \frac{4}{m} W_m$ is an unbiased estimate for π (i.e. $E[W'_m] = \pi$)
- How many samples do we need to obtain a good estimate?

$$\Pr(|W'_m - \pi| \geq \epsilon) = ?$$

Example: Estimating a Parameter

- Evaluating the probability that a particular DNA mutation occurs in the population.
- Given a DNA sample, a lab test can determine if it carries the mutation.
- The test is expensive and we would like to obtain a relatively reliable estimate from a minimum number of samples.
- p = the unknown value;
- n = number of samples, $\tilde{p}n$ had the mutation.
- Given sufficient number of samples we expect the value p to be in the neighborhood of sampled value \tilde{p} , but we cannot predict any single value with high confidence.

Confidence Interval

Instead of predicting a single value for the parameter we give an *interval* that is *likely* to contain the parameter.

Definition

A $1 - q$ **confidence interval** for a parameter T is an interval $[\tilde{p} - \delta, \tilde{p} + \delta]$ such that

$$\Pr(T \in [\tilde{p} - \delta, \tilde{p} + \delta]) \geq 1 - q.$$

We want to minimize 2δ and q , with minimum n .

Using $\tilde{p}n$ as our estimate for pn , we need to compute δ and q such that

$$\Pr(p \in [\tilde{p} - \delta, \tilde{p} + \delta]) = \Pr(np \in [n(\tilde{p} - \delta), n(\tilde{p} + \delta)]) \geq 1 - q.$$

- The random variable here is the interval $[\tilde{p} - \delta, \tilde{p} + \delta]$ (or the value \tilde{p}), while p is a fixed (unknown) value.
- $n\tilde{p}$ has a binomial distribution with parameters n and p , and $\mathbf{E}[\tilde{p}] = p$. If $p \notin [\tilde{p} - \delta, \tilde{p} + \delta]$ then we have one of the following two events:
 - ① If $p < \tilde{p} - \delta$, then $n\tilde{p} \geq n(p + \delta) = np \left(1 + \frac{\delta}{p}\right)$, or $n\tilde{p}$ is larger than its expectation by a $\frac{\delta}{p}$ factor.
 - ② If $p > \tilde{p} + \delta$, then $n\tilde{p} \leq n(p - \delta) = np \left(1 - \frac{\delta}{p}\right)$, and $n\tilde{p}$ is smaller than its expectation by a $\frac{\delta}{p}$ factor.

$$\begin{aligned}
& \Pr(p \notin [\tilde{p} - \delta, \tilde{p} + \delta]) \\
&= \Pr\left(n\tilde{p} \leq np\left(1 - \frac{\delta}{p}\right)\right) + \Pr\left(n\tilde{p} \geq np\left(1 + \frac{\delta}{p}\right)\right) \\
&\leq e^{-\frac{1}{2}np\left(\frac{\delta}{p}\right)^2} + e^{-\frac{1}{3}np\left(\frac{\delta}{p}\right)^2} \\
&= e^{-\frac{n\delta^2}{2p}} + e^{-\frac{n\delta^2}{3p}}.
\end{aligned}$$

But the value of p is unknown, A simple solution for the case of estimating π is to use the fact that $p = \pi/4 \leq 1$ to prove

$$\Pr(p \notin [\tilde{p} - \delta, \tilde{p} + \delta]) \leq e^{-\frac{n\delta^2}{2}} + e^{-\frac{n\delta^2}{3}}.$$

Setting $q = e^{-\frac{n\delta^2}{2}} + e^{-\frac{n\delta^2}{3}}$, we obtain a tradeoff between δ , n , and the error probability q .

$$q = e^{-\frac{n\delta^2}{2}} + e^{-\frac{n\delta^2}{3}}$$

If we want to obtain a $1 - q$ confidence interval $[\tilde{p} - \delta, \tilde{p} + \delta]$,

$$n \geq \frac{3}{\delta^2} \ln \frac{2}{q}$$

samples are enough.

Set Balancing

Given an $n \times n$ matrix \mathcal{A} with entries in $\{0, 1\}$, let

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ \dots \\ b_n \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ \dots \\ c_n \end{pmatrix}.$$

Find a vector \bar{b} with entries in $\{-1, 1\}$ that minimizes

$$\|\mathcal{A}\bar{b}\|_{\infty} = \max_{i=1, \dots, n} |c_i|.$$

Theorem

For a random vector \bar{b} , with entries chosen independently and with equal probability from the set $\{-1, 1\}$,

$$\Pr(\|\mathcal{A}\bar{b}\|_\infty \geq \sqrt{4n \ln n}) \leq \frac{2}{n}.$$

The $\sum_{i=1}^n a_{j,i} b_i$ (excluding the zero terms) is a sum of independent $-1, 1$ random variable. We need a bound on such sum.

Chernoff Bound for Sum of $\{-1, +1\}$ Random Variables

Theorem

Let X_1, \dots, X_n be independent random variables with

$$\Pr(X_i = 1) = \Pr(X_i = -1) = \frac{1}{2}.$$

Let $X = \sum_1^n X_i$. For any $a > 0$,

$$\Pr(X \geq a) \leq e^{-\frac{a^2}{2n}}.$$

de Moivre – Laplace approximation: For any k , such that $|k - np| \leq a$

$$\binom{n}{k} p^k (1-p)^{n-k} \approx \frac{1}{\sqrt{2\pi np(1-p)}} e^{-\frac{a^2}{2np(1-p)}}$$

For any $t > 0$,

$$\mathbf{E}[e^{tX_i}] = \frac{1}{2}e^t + \frac{1}{2}e^{-t}.$$

$$e^t = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^i}{i!} + \cdots$$

and

$$e^{-t} = 1 - t + \frac{t^2}{2!} + \cdots + (-1)^i \frac{t^i}{i!} + \cdots$$

Thus,

$$\begin{aligned} \mathbf{E}[e^{tX_i}] &= \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \sum_{i \geq 0} \frac{t^{2i}}{(2i)!} \\ &\leq \sum_{i \geq 0} \frac{\left(\frac{t^2}{2}\right)^i}{i!} = e^{t^2/2} \end{aligned}$$

$$\mathbf{E}[e^{tX}] = \prod_{i=1}^n \mathbf{E}[e^{tX_i}] \leq e^{nt^2/2},$$

$$Pr(X \geq a) = Pr(e^{tX} > e^{ta}) \leq \frac{\mathbf{E}[e^{tX}]}{e^{ta}} \leq e^{t^2n/2 - ta}.$$

Setting $t = a/n$ yields

$$Pr(X \geq a) \leq e^{-\frac{a^2}{2n}}.$$

By symmetry we also have

Corollary

Let X_1, \dots, X_n be independent random variables with

$$\Pr(X_i = 1) = \Pr(X_i = -1) = \frac{1}{2}.$$

Let $X = \sum_{i=1}^n X_i$. Then for any $a > 0$,

$$\Pr(|X| > a) \leq 2e^{-\frac{a^2}{2n}}.$$

Application: Set Balancing

Theorem

For a random vector \bar{b} , with entries chosen independently and with equal probability from the set $\{-1, 1\}$,

$$\Pr(\|\mathcal{A}\bar{b}\|_\infty \geq \sqrt{4n \ln n}) \leq \frac{2}{n} \quad (6)$$

- Consider the i -th row $\bar{a}_i = a_{i,1}, \dots, a_{i,n}$.
- Let k be the number of 1's in that row.
- $Z_i = \sum_{j=1}^k a_{i,j} b_j$.
- If $k \leq \sqrt{4n \ln n}$ then clearly $Z_i \leq \sqrt{4n \ln n}$.

If $k > \sqrt{4n \log n}$, the k non-zero terms in the sum Z_i are independent random variables, each with probability $1/2$ of being either $+1$ or -1 .

Using the Chernoff bound:

$$\Pr \left\{ |Z_i| > \sqrt{4n \log n} \right\} \leq 2e^{-4n \log n / (2k)} \leq 2e^{-4n \log n / (2n)} \leq \frac{2}{n^2},$$

where we use the fact that $n \geq k$.

The result follows by union bound on the n rows.

Hoeffding's Inequality

Large deviation bound for more general random variables:

Theorem (Hoeffding's Inequality)

Let X_1, \dots, X_n be independent random variables such that for all $1 \leq i \leq n$, $E[X_i] = \mu$ and $\Pr(a \leq X_i \leq b) = 1$. Then

$$\Pr\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - \mu\right| \geq \epsilon\right) \leq 2e^{-2n\epsilon^2/(b-a)^2}$$

Lemma

(Hoeffding's Lemma) Let X be a random variable such that $\Pr(X \in [a, b]) = 1$ and $E[X] = 0$. Then for every $\lambda > 0$,

$$E[e^{\lambda X}] \leq e^{\lambda^2(a-b)^2/8}.$$

Proof of the Lemma

Since $f(x) = e^{\lambda x}$ is a convex function, for any $\alpha \in (0, 1)$ and $x \in [a, b]$,

$$f(X) \leq \alpha f(a) + (1 - \alpha)f(b).$$

Thus, for $\alpha = \frac{b-x}{b-a} \in (0, 1)$,

$$e^{\lambda x} \leq \frac{b-x}{b-a} e^{\lambda a} + \frac{x-a}{b-a} e^{\lambda b}.$$

Taking expectation, and using $\mathbf{E}[X] = 0$, we have

$$\mathbf{E}[e^{\lambda X}] \leq \frac{b}{b-a} e^{\lambda a} + \frac{a}{b-a} e^{\lambda b} \leq e^{\lambda^2(b-a)^2/8}.$$

Proof of the Bound

Let $Z_i = X_i - \mathbf{E}[X_i]$ and $Z = \frac{1}{n} \sum_{i=1}^n X_i$.

$$\Pr(Z \geq \epsilon) \leq e^{-\lambda\epsilon} \mathbf{E}[e^{\lambda Z}] \leq e^{-\lambda\epsilon} \prod_{i=1}^n \mathbf{E}[e^{\lambda X_i/n}] \leq e^{-\lambda\epsilon + \frac{\lambda^2(b-a)^2}{8n}}$$

Set $\lambda = \frac{4n\epsilon}{(b-a)^2}$ gives

$$\Pr\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - \mu\right| \geq \epsilon\right) = \Pr(Z \geq \epsilon) \leq 2e^{-2n\epsilon^2/(b-a)^2}$$

A More General Version

Theorem

Let X_1, \dots, X_n be independent random variables with $\mathbf{E}[X_i] = \mu_i$ and $\Pr(B_i \leq X_i \leq B_i + c_i) = 1$, then

$$\Pr\left(\left|\sum_{i=1}^n X_i - \sum_{i=1}^n \mu_i\right| \geq \epsilon\right) \leq e^{-\frac{2\epsilon^2}{\sum_{i=1}^n c_i^2}}$$

Application: Job Completion

We have n jobs, job i has expected run-time μ_i . We terminate job i if it runs $\beta\mu_i$ time. When will the machine will be free of jobs?

X_i = execution time of job i . $0 \leq X_i \leq \beta\mu_i$.

$$\Pr\left(\left|\sum_{i=1}^n X_i - \sum_{i=1}^n \mu_i\right| \geq \epsilon \sum_{i=1}^n \mu_i\right) \leq 2e^{-\frac{2\epsilon^2(\sum_{i=1}^n \mu_i)^2}{\sum_{i=1}^n \beta^2 \mu_i^2}}$$

Assume all $\mu_i = \mu$

$$\Pr\left(\left|\sum_{i=1}^n X_i - n\mu\right| \geq \epsilon n\mu\right) \leq 2e^{-\frac{2\epsilon^2 n^2 \mu^2}{n\beta^2 \mu^2}} = 2e^{-2\epsilon^2 n/\beta^2}$$

Let $\epsilon = \beta\sqrt{\frac{\log n}{n}}$, then

$$\Pr\left(\left|\sum_{i=1}^n X_i - n\mu\right| \geq \beta\mu\sqrt{n \log n}\right) \leq 2e^{-\frac{2\beta^2 \mu^2 n \log n}{n\beta^2 \mu^2}} = \frac{2}{n^2}$$