Linear Algebra
Vectors

- A vector is a **magnitude** and a **direction**
- **Magnitude** = $||\mathbf{v}||$
  - Also known as **norm, length**
- **Direction**
  - Represented by **unit vectors** (vectors with a length of 1 that point along distinct axes)
  - Often denoted by a hat above the letter ($\hat{x}$)
  - $\hat{x} = \frac{x}{||x||}$
- **A vector is not a position**
  - A position is a distinct point
  - A vector travels from one point to another
Vector addition

- Vector addition in $\mathbb{R}^1$
  - Familiar addition of real numbers

- Vector addition in $\mathbb{R}^2$
  - Component-wise addition
  - Result, $\mathbf{v}_1 + \mathbf{v}_2$, plotted in $\mathbb{R}^2$ is the new vector

\[ \mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix} \]
Adding vectors visually

- $\mathbf{v}_2$ added to $\mathbf{v}_1$, using the parallelogram rule
  - reposition $\mathbf{v}_2$ so that its tail is at the head of vector $\mathbf{v}_1$
  - define $\mathbf{v}_1 + \mathbf{v}_2$ as the head of the new vector

- Or equivalently, add $\mathbf{v}_1$ to $\mathbf{v}_2$
Vector subtraction

- Same as adding but one of the vectors is flipped

- Ex. Find \((8, 9) - (2, 4)\)
  - Flip \((2, 4)\) so that the tail is at the head of \((8, 9)\)
  - Effectively \((8, 9) + (-2, -4) = (6, 5)\)
Vector scalar multiplication

- Geometrically, think of scalar multiplication as **scaling** the vector
  - Magnitude is multiplied
  - Direction is not affected (unless zeroed or negated)

- Scaled vectors are **always parallel**
  - Scaled components yield same slope
Linear dependence

- Set of all scalar multiples of a vector is a line through the origin
- Two vectors are \textbf{linearly dependent} when one is a multiple of the other
  - On the same line
- Definition of dependence for three or more vectors is trickier
  - Geometric intuition:
    - In $\mathbb{R}^3$ when on the same plane
    - In $\mathbb{R}^n$ when on the same n-dimensional hyperplane
Linear dependence

Plot of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

Rotated to see the three vectors on same plane
Standard basis vectors

- The unit vectors (i.e. whose length is 1) on the x and y-axes are called the **standard basis vectors**
- The collection of scalar multiples of \[
\begin{bmatrix} 
1 \\
0 
\end{bmatrix}
\]
gives the first coordinate axis
- The collection of scalar multiples of \[
\begin{bmatrix} 
0 \\
1 
\end{bmatrix}
\]
gives the second coordinate axis
Basis vectors

- Any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ can be expressed as the sum of scalar multiples of the unit vectors:
  \[
  \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}
  \]

- We call these two vectors \textit{basis vectors} for $\mathbb{R}^2$ because any other vector can be expressed in terms of them
  - Very important concept
  - Can you guess what the standard basis vectors for $\mathbb{R}^3$ are?
Non-orthogonal basis vectors

- \([1 \ 0] \) and \([0 \ 1] \) are perpendicular. Is this necessary?

- Can we make any vector \([n \ m] \) from scalar multiples of random vectors \([a \ b] \) and \([c \ d] \)?

- Is there a solution for all \(n, m\) to the following, where \(\alpha\) and \(\beta\) are unknown?

\[
\begin{bmatrix} n \\ m \end{bmatrix} = \alpha \begin{bmatrix} a \\ b \end{bmatrix} + \beta \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \alpha a + \beta c \\ \alpha b + \beta d \end{bmatrix}
\]
Non-orthogonal basis vectors

- Is there a solution to the following, where $\alpha$ and $\beta$ are unknown:

$$\begin{bmatrix} n \\ m \end{bmatrix} = \alpha \begin{bmatrix} a \\ b \end{bmatrix} + \beta \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \alpha a + \beta c \\ \alpha b + \beta d \end{bmatrix}$$

- Just a linear system of two equations. When is this unsolvable? How does it make sense geometrically?

![Diagram showing different scenarios for the solution](image-url)
Dot product

- Operation on two vectors

\[
\begin{bmatrix}
  a \\
  b
\end{bmatrix} \cdot \begin{bmatrix}
  x \\
  y
\end{bmatrix} = ax + by
\]

- Result is a scalar!
  - Also known as scalar product or inner product

\[
\begin{bmatrix}
  a \\
  b \\
  c
\end{bmatrix} \cdot \begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} = ax + by + cz
\]

- Defined for vectors of any size
  - Both vectors **must** have the same size

- Dot products instructions are used in SIMD registers
Calculating magnitude using dot product

- Let’s find the magnitude of vector $\mathbf{v}$ from origin to house
  \[ \mathbf{v} = (2, 4) - (0, 0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \]

- Use Pythagorean theorem (**2D only**)
  \[ ||\mathbf{v}|| = \sqrt{2^2 + 4^2} = \sqrt{20} \]

- Use **dot product** (**All dimensions**)!
  \[ \mathbf{v} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2^2 + 4^2 \]
  \[ ||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \]
Find the angle between two vectors

- Let’s find the angle between \( \mathbf{v} \) and x-axis

\[
\mathbf{v} = (2, 4) - (0, 0) = [2 \\ 4]
\]

\[
\mathbf{x} = [1 \\ 0]
\]

- Angle between two vectors is \( \theta \), where

\[
\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \cos (\theta)
\]

- Proof is omitted here but try it at home!
- For our problem it would be solving for \( \theta \) in:

\[
\mathbf{v} \cdot \mathbf{x} = ||\mathbf{v}|| ||\mathbf{x}|| \cos \theta
\]
Determining right angles

- Angle between two vectors is $\theta$, where

$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta)$

- If $\theta$ is a right angle (i.e. $\theta=90^\circ, \pi/2$), $\cos(\theta) = 0$
- So if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular,

$\mathbf{a} \cdot \mathbf{b} = 0$
Projections

- Let’s find projection of $v$ onto $x$-axis

\[
v = (2, 4) - (0, 0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}
\]

\[x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\]

\[
proj_x(v) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}
\]
Projections

- Projections in general
  - Recall $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\theta)$

- Use Cases?
  - We’ll use projections when we create a synthetic camera in our Camtrans lab for the Sceneview project

- In mathematical terms, the projection of vector $\mathbf{a}$ onto vector $\mathbf{b}$
  - Length times unit vector in direction of $\mathbf{b}$
    $$\text{proj}_b(\mathbf{a}) = \|\mathbf{a}\| \cos \theta \cdot \frac{\mathbf{b}}{\|\mathbf{b}\|}$$

- And the length of the projection
  $$\|\text{proj}_b(\mathbf{a})\| = \|\mathbf{a}\| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}$$
Cross product

- **Operation on two 3D vectors**
  - Two vectors define a plane
- **Output is a vector!**
  - Follows right-hand rule
  - Result is a vector perpendicular to plane defined by input vectors
  - Its magnitude is equal to the area of the parallelogram formed by the two vectors
- **In terms of vector components, the cross product is calculated**
  
  \[
  \begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
  \end{bmatrix} \times \begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  \end{bmatrix} = \begin{bmatrix}
  a_2b_3 - a_3b_2 \\
  a_3b_1 - a_1b_3 \\
  a_1b_2 - a_2b_1 \\
  \end{bmatrix}
  \]

  - Note: vectors can be any arbitrary non-collinear pair, don't have to be in xy plane

**Use Cases?**
- Finding normals for our lighting model
Matrix

- A matrix is a rectangular array of numbers or other mathematical objects.
- Can operate on with addition (matrix-matrix) and multiplication (scalar, vector, and matrix).
- Size: $m \times n \rightarrow \text{rows} \times \text{cols}$

Notation:

$$
A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
$$

- For the purposes of this class, matrices are used to represent transformations, which act on points, vectors, and other matrices.
Product of row vector and column vector

- We haven’t stressed this so far but there is distinction between vector written as row vs. vector written as column
  - Use column vectors usually
- Product of a row vector and column vector denoted
  - Vectors **must** be the same size
- Result is actually **dot product** of these two vectors:

\[
\begin{bmatrix}
5 & 3 & 6
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}
= 1 \cdot 5 + 0 \cdot 3 + 1 \cdot 6 = 11
\]
Matrix and vector multiplication

- Matrix-vector multiplication produces a new vector.
  - \( \mathbf{v}' = M\mathbf{v} \)
  - Both \( \mathbf{v} \) and \( \mathbf{v}' \) column vectors
- kth element of \( \mathbf{v}' \) is the product of kth row of \( M \) with \( \mathbf{v} \)
  - Same as the dot product of kth row of \( M \) with \( \mathbf{v} \)
- If \( M \) is an \( m \times n \) matrix, \( \mathbf{v} \) must be length \( n \)
  - Otherwise dot product not defined
  - \((m \times n) (n \times 1) \rightarrow m \times 1\)

Use Cases?
- Transform vertices into world space / camera space / screen space
- Transforming normal vectors (more later)
- Moving and animating graphical models

\[
\mathbf{v}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = M\mathbf{v}
\]

where \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}, \mathbf{v}' = \begin{bmatrix} x' \\ y' \end{bmatrix} \)
Matrix and matrix multiplication

- $MN_{ij} =$ the dot product of the $i$th row of $M$ and the $j$th column of $N$
  - Product of $i$th row vector of $M$ and $j$th column vector of $N$
  - Subscript denotes row, then column
- If $M$ is an $m \times n$ matrix, then $N$ must be an $n \times k$ matrix.
  - Otherwise, dot product of $i$th row and $j$th column would not make sense
- $MN$ is an $m \times k$ matrix
- Order of multiplication matters!
  - In general $MN \neq NM$
  - $NM$ may not even be defined!
Matrix multiplication identity

- Scalar operations like addition and multiplication have identities
  - $a + 0 = a$
  - $a \times 1 = a$
  - 0 for addition, 1 for multiplication

- What is the identity for matrix multiplication?

- The identity matrix $I$ is a matrix with all zeros, except for on the diagonal, where it is all ones

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}_{2 \times 2}
\quad
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}_{3 \times 3}
\]

- Identity matrix exists for different dimensions

- Multiplication of an object (matrix or vector) by identity produces original object
Properties of Matrix Multiplication

- Properties similar to scalar multiplication
- Associative property
  - \((AB)C = A(BC)\)
  - Important for combining transformations -- we can combine and group adjacent
- Distributive property
  - \(A(u + v) = Au + Av\) \([\text{vectors}]\)
  - Use transformation of component parts to transform complicated vector
- Identity
  - \(AI = A = IA\)
- Inverse (more later)
  - For some \(A\), there is a matrix \(A^{-1}\) such that
  - \(AA^{-1} = I = A^{-1}A\)
Visual Matrix Multiplication

- Since matrix multiplication is distributive over vectors, we can tell where each vector will be sent by knowing where each standard basis vector “is sent”
- Originally, we have a unit grid
- Then we apply transformation and set up a grid based on images of two standard basis vectors
Visual Matrix Multiplication

- Now to see where \[
\begin{bmatrix}
2 \\
3
\end{bmatrix}
\] ends, instead of going 2 steps along x-axis, and 3 along y-axis, go 2 steps along image of \[
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\] and 3 along image of \[
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\]
Inverse of a Matrix

- A and B are inverses if
  \[ AB = BA = I \]
- Some matrices don’t have an inverse matrix
- Matrices for rotation, translation, and scaling have inverses

What does an Inverse do?

- The inverse \( A^{-1} \) of transformation A will “undo” the result of transforming by A
- If A scales uniformly by a factor of 2 and then rotates 135 degrees, then \( A^{-1} \) will rotate by \(-135\) degrees and then scales uniformly by \( \frac{1}{2} \)
- Note that A is a composite matrix!
Composite Inverses

- Why does it matter if you scale or rotate first?
  - Results would be the same in this example but that may not always be the case
- Recall slide 32 from the Transformations lecture
- Inverses must apply the inverse transformations in the reverse order else you can get an incorrect result

\[ A = M_1 M_2 \]
\[ A^{-1} = (M_1 M_2)^{-1} = M_2^{-1} M_1^{-1} \]
How to get the inverse

- `glm::inverse()`
- Use Gauss-Jordan elimination
- To find $A^{-1}$ with Gauss-Jordan elimination
  - Augment $A$ with $I$ to get $[A|I]$
  - Reduce the new matrix into reduced row echelon form (`rref`)
  - The matrix is now in the form $[I|A^{-1}]$

To reduce a matrix into `rref` we are allowed to perform any of the three elementary row operations. These are:

- Multiply a row by a nonzero constant
- Interchange two rows
- Add a multiple of one row to another row

\[
\begin{bmatrix}
3 & 0 & 2 & 1 & 0 & 0 \\
2 & 0 & -2 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 \\
0 & 1 & 0 & \frac{1}{5} & \frac{3}{10} & 1 \\
0 & 0 & 1 & \frac{1}{5} & -\frac{3}{10} & 0
\end{bmatrix}
\]
Example

- This will be a small example because doing this by hand takes a while. Let’s find $A^{-1}$ for:

\[
A = \begin{bmatrix} 11 & 13 \\ 17 & 19 \end{bmatrix}
\]

- Augment this with the identity to get

\[
[A | I] = \begin{bmatrix} 11 & 13 & 1 & 0 \\ 17 & 19 & 0 & 1 \end{bmatrix}
\]
Example

- Row operation 1, multiply row 1 by $1/11$.

\[
\begin{bmatrix}
1 & \frac{13}{11} & \frac{1}{11} & 0 \\
\frac{17}{11} & \frac{19}{11} & 0 & 1 \\
\end{bmatrix}
\]

- Row operation 3, multiply row 1 by $-17$ and add it to row 2

\[
\begin{bmatrix}
1 & \frac{13}{11} & \frac{1}{11} & 0 \\
0 & -\frac{12}{11} & -\frac{17}{11} & 1 \\
\end{bmatrix}
\]
• Row operation 1, multiply row 2 by $-11/12$

$$
\begin{bmatrix}
1 & 13/11 & 1/11 & 0 \\
0 & 1 & 17/12 & -11/12
\end{bmatrix}
$$

• Row operation 3, multiply row 2 by $-13/11$ and add to row 1

$$
[U|A^{-1}] =
\begin{bmatrix}
1 & 0 & -19/12 & 13/12 \\
0 & 1 & 17/12 & -11/12
\end{bmatrix}
$$

• Therefore:

$$
A^{-1} =
\begin{bmatrix}
-19/12 & 13/12 \\
17/12 & -11/12
\end{bmatrix}
$$
Additional Insight into the Gauss-Jordan Elimination

- When do we need to use this method?
  - A lot of matrices inverses can be deduced without applying this process
  - For example, to invert normalizing transformation, the computer will apply inversion method

- Why does this method work?
  - Algebraic intuition: row operations are equivalent of operations on system of equations
  - System of equations invariant under these operations

- Another explanation
  - Row operations are actually matrix multiplications
  - Check this link out if interested: http://aleph0.clarku.edu/~djoyce/ma130/elementary.pdf
Questions?