Bayes’ Rule
Monty Hall Problem

"Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the other doors, opens another door, say No. 3, which has a goat. He then says to you, 'Do you want to pick door No. 2?' Is it to your advantage to take the switch?"

Marilyn vos Savant

Image Source
Derivation

Facts

- $P(A \mid B) = P(A \text{ and } B) / P(B)$ implies $P(A \text{ and } B) = P(A \mid B) P(B)$
- $P(B \mid A) = P(A \text{ and } B) / P(A)$ implies $P(A \text{ and } B) = P(B \mid A) P(A)$
- $P(A \mid B) P(B) = P(B \mid A) P(A)$ implies $P(A \mid B) = P(B \mid A) P(A) / P(B)$
- $P(A \mid B) P(B) = P(B \mid A) P(A)$ implies $P(B \mid A) = P(A \mid B) P(B) / P(A)$

Rule

$$p(B \mid A) = \frac{p(A \mid B)p(B)}{p(A)}$$

In words, if we know one conditional probability, then we can use Bayes’ rule to find the opposite conditional probability.
Derivation

Facts

- $P(A \mid B) = P(A \text{ and } B) / P(B)$ implies $P(A \text{ and } B) = P(A \mid B) \cdot P(B)$
- $P(B \mid A) = P(A \text{ and } B) / P(A)$ implies $P(A \text{ and } B) = P(B \mid A) \cdot P(A)$
- $P(A \mid B) \cdot P(B) = P(B \mid A) \cdot P(A)$ implies $P(A \mid B) = P(B \mid A) \cdot P(A) / P(B)$
- $P(B) = P(B \text{ and } A) + P(B \text{ and } A^c) = P(B \mid A) \cdot P(A) + P(B \mid A^c) \cdot P(A^c)$

Rule

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B \mid A) \cdot P(A) + P(B \mid A^c) \cdot P(A^c)}$$

In words, if we know one conditional probability, then we can use Bayes’ rule to find the opposite conditional probability.
There are 175 million adults in the United States. Suppose some rare disease affects 1 in 100,000 of us. Further suppose that we can screen for this disease. Given that someone tests positive, how likely is it that they have the disease? **Spoiler:** Surprisingly low!
Worked Example 1

Assumptions in English

- The disease affects 1 in 100,000.
- If someone has the disease, the screen always flags them.
- If someone does not have the disease, the screen still flags them 1 in 10,000 times.

Assumptions as Probabilities

- $P(\text{Diseased}) = 0.00001$
- $P(\text{Test Positive} | \text{Diseased}) = 1$
- $P(\text{Test Positive} | \text{Healthy}) = 0.0001$
Worked Example 1

\[
P(D \text{ and } T^+) = 0.00001
\]
\[
P(D \text{ and } T^-) = 0
\]
\[
P(H \text{ and } T^+) = 0.0001
\]
\[
P(H \text{ and } T^-) = 0.99989
\]

\(T^+: 1\)
\(T^-: 0\)

\(D: 0.00001\)
\(H: 0.99999\)

\(T^+: 0.0001\)
\(T^-: 0.9999\)
Worked Example 1

Bayes Rule

\[
P(D^+ | T^+) = \frac{P(T^+ | D^+) \cdot P(D^+)}{P(T^+)}
\]

\[
= (1) \cdot (0.00001) / (0.00001 + 0.0001)
\]

\[
= 0.00001 / 0.00011
\]

\[
\approx 0.09
\]

If someone tests positive, they have about a 9% chance of actually having the disease.
There are 250 million adults in the United States. The government monitors them all (us!) to detect terrorism. Given that someone is suspected of terrorism, what is the probability that the person is a terrorist? **Spoiler:** Surprisingly low!
Worked Example 2

Assumptions in English

- There are 250 million adults in the United States, of whom 250 are terrorists.
- If someone is a terrorist, then the government’s monitoring system will always flag them.
- If someone is not a terrorist, then the system will flag them anyway 1 in 10,000 times.

Assumptions as Probabilities

- $P(T) = 0.000001$
- $P(S+|T) = 1$
- $P(S+|NT) = 0.0001$
Worked Example 2

\[
\begin{align*}
P(T \text{ and } S^+) &= 0.000001 \\
P(T \text{ and } S^-) &= 0 \\
P(NT \text{ and } S^+) &= 0.0001 \\
P(NT \text{ and } S^-) &= 0.99989
\end{align*}
\]
Worked Example 2

If someone is suspected of terrorism, there is about a 1% chance they are actually a terrorist.

\[
P(T \text{ and } S^+) = 0.000001 \\
P(T \text{ and } S^-) = 0 \\
P(NT \text{ and } S^+) = 0.0001 \\
P(NT \text{ and } S^-) = 0.99989
\]

\[
P(D^+ | T^+) = 1 \times 0.000001 / (0.000001 + 0.0001) \\
\approx 0.01
\]

Bayes Rule
There are 250 million adults in the United States. The government monitors them all (us!) to detect terrorism. Given that someone is convicted of terrorism, what is the probability that the person is a terrorist? **Spoiler:** Higher, but arguably not high enough!
Worked Example 3

Assumptions in English:

- There are 250 million adults in the United States, of whom 250 are terrorists.
- If someone is a terrorist, then the government’s monitoring system will always flag them.
- If someone is not a terrorist, then the system will flag them anyway 1 in 10,000 times.
- The criminal justice system makes an error 1% of the time:
  - A terrorist is innocent 1% of the time.
  - A flagged civilian is guilty 1% of the time.

Assumptions as Probabilities:

- \( P(T) = 0.000001 \)
- \( P(S+|T) = 1 \)
- \( P(S+|NT) = 0.0001 \)
- \( P(C|S+ \text{ and } T) = .99, \ P(C|S+ \text{ and } NT) = .01 \)
Worked Example 3

If someone is convicted of terrorism, there is only about a 49.7% chance they are actually a terrorist!