Quiz

- Restate equations

$$\begin{aligned}\mathbf{v}_0 &= \mathbf{v}_0^* \\ \mathbf{v}_1 &= \alpha_{01}\mathbf{v}_0^* + \mathbf{v}_1 \\ \mathbf{v}_2 &= \alpha_{02}\mathbf{v}_0^* + \alpha_{12}\mathbf{v}_1^* + \mathbf{v}_2 \\ \mathbf{v}_3 &= \alpha_{03}\mathbf{v}_0^* + \alpha_{13}\mathbf{v}_1^* + \alpha_{23}\mathbf{v}_2^* + \mathbf{v}_3^*\end{aligned}$$

  as a single matrix equation: (some matrix) = (some other matrix) (yet another matrix)
- What is special about the two matrices on the right-hand side?
- Let $\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_3^*$ be mutually orthogonal vectors. Assume in addition that they are nonzero. You are to show that they are linearly independent. That is, if $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ satisfy the equation

$$\mathbf{0} = \alpha_0\mathbf{v}_0^* + \alpha_1\mathbf{v}_1^* + \alpha_2\mathbf{v}_2^* + \alpha_3\mathbf{v}_3^*$$

  then $\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$.
  For purposes of this proof, it suffices to show that $\alpha_0 = 0$.

Matrix form for `orthogonalize`

For `project_orthogonal`, we had

$$\begin{bmatrix} \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_0 & \cdots & \mathbf{v}_n & \mathbf{b}^\perp \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_n \\ 1 \end{bmatrix}$$

For `orthogonalize`, we have

$$\begin{bmatrix} \mathbf{v}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_0^* \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{v}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_0^* & \mathbf{v}_1^* \end{bmatrix} \begin{bmatrix} \alpha_{01} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_0^* & \mathbf{v}_1^* & 2\mathbf{v}_2^* \end{bmatrix} \begin{bmatrix} \alpha_{02} \\ \alpha_{12} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \mathbf{v}_3^* \end{bmatrix} \begin{bmatrix} \alpha_{03} \\ \alpha_{13} \\ \alpha_{23} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \mathbf{v}_3^* \end{bmatrix} \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & \alpha_{03} \\ & 1 & \alpha_{12} & \alpha_{13} \\ & & 1 & \alpha_{23} \\ & & & 1 \end{bmatrix}$$

Mutually orthogonal nonzero vectors are linearly independent

Proposition: Mutually orthogonal nonzero vectors are linearly independent.

Proof: Let $\mathbf{v}_0^*, \mathbf{v}_1^*, \ldots, \mathbf{v}_n^*$ be mutually orthogonal nonzero vectors.
Suppose $\alpha_0, \alpha_1, \ldots, \alpha_n$ are coefficients such that

$$\mathbf{0} = \alpha_0\,\mathbf{v}_0^* + \alpha_1\,\mathbf{v}_1^* + \cdots + \alpha_n\,\mathbf{v}_n^*$$

We must show that therefore the coefficients are all zero.
To show that α_0 is zero, take inner product with $\mathbf{v}_0^*$ on both sides:

$$\begin{aligned}
\langle \mathbf{v}_0^*, \mathbf{0} \rangle &= \langle \mathbf{v}_0^*, \alpha_0\,\mathbf{v}_0^* + \alpha_1\,\mathbf{v}_1^* + \cdots + \alpha_n\,\mathbf{v}_n^* \rangle \\
&= \alpha_0\ \langle \mathbf{v}_0^*, \mathbf{v}_0^* \rangle + \alpha_1\ \langle \mathbf{v}_0^*, \mathbf{v}_1^* \rangle + \cdots + \alpha_n\ \langle \mathbf{v}_0^*, \mathbf{v}_n^* \rangle \\
&= \alpha_0 \|\mathbf{v}_0^*\|^2 + \alpha_1\,0 + \cdots + \alpha_n\,0 \\
&= \alpha_0 \|\mathbf{v}_0^*\|^2
\end{aligned}$$

The inner product $\langle \mathbf{v}_0^*, 0 \rangle$ is zero, so $\alpha_0\,\|\mathbf{v}_0^*\|^2 = 0$. Since $\mathbf{v}_0^*$ is nonzero, its norm is nonzero, so the only solution is $\alpha_0 = 0$.
Can similarly show that $\alpha_1 = \cdots = \alpha_n = 0$. QED

Computing a basis

Proposition: Mutually orthogonal nonzero vectors are linearly independent.

What happens if we call the `orthogonalize` procedure on a list `vlist`$=[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ of vectors that are linearly dependent?

$\dim \text{Span } \{\mathbf{v}_0, \ldots, \mathbf{v}_n\} < n+1$.

`orthogonalize(`$[\mathbf{v}_0, \ldots, \mathbf{v}_n]$`)` returns $[\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*]$

The vectors $\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*$ are mutually orthogonal.

They can't be linearly independent since they span a space of dimension less than $n+1$.

Therefore some of them must be zero vectors.

Leaving out the zero vectors does not change the space spanned...

Let S be the subset of $\{\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*\}$ consisting of nonzero vectors.

$\text{Span } S = \text{Span } \{\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*\} = \text{Span } \{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$

Proposition implies that S is linearly independent.

Thus S is a basis for $\text{Span } \{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$.

Computing a basis

Therefore in principle the following algorithm computes a basis for Span $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$:

def find_basis($[\mathbf{v}_0, \ldots, \mathbf{v}_n]$):
    "Return the list of nonzero starred vectors."
    $[\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*] = \texttt{orthogonalize}([\mathbf{v}_0, \ldots, \mathbf{v}_n])$
    return $[\mathbf{v}^*$ for $\mathbf{v}^*$ in $[\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*]$ if $\mathbf{v}^*$ is not the zero vector$]$

Example:

Suppose $\texttt{orthogonalize}([\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6])$ returns $[\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_3^*, \mathbf{v}_4^*, \mathbf{v}_5^*, \mathbf{v}_6^*]$ and the vectors $\mathbf{v}_2^*$,$\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero.

Then the remaining output vectors $\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*$ form a basis for Span $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$.

Recall **Lemma:** Every finite set T of vectors contains a subset S that is a basis for Span T.

What about finding a subset of $\mathbf{v}_0, \ldots, \mathbf{v}_n$ that is a basis?

Proposed algorithm:[

def find_subset_basis($[\mathbf{v}_0, \ldots, \mathbf{v}_n]$):
    "Return the list of original vectors that correspond to nonzero starred vectors."

Computing a basis

Therefore in principle the following algorithm computes a basis for Span $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$:

```
def find_basis([v_0, ..., v_n]):
    "Return the list of nonzero starred vectors."
    [v*_0, ..., v*_n] = orthogonalize([v_0, ..., v_n])
    return [v* for v* in [v*_0, ..., v*_n] if v* is not the zero vector]
```

Recall **Lemma:** Every finite set T of vectors contains a subset S that is a basis for Span T.

What about finding a subset of $\mathbf{v}_0, \ldots, \mathbf{v}_n$ that is a basis?

Proposed algorithm:[

```
def find_subset_basis([v_0, ..., v_n]):
    "Return the list of original vectors that correspond to nonzero starred vectors."
    [v*_0, ..., v*_n] = orthogonalize([v_0, ..., v_n])
    Return [v_i for i in {0, ..., n} if v*_i is not the zero vector]
```

Is this correct?

Correctness of find_subset_basis

def find_subset_basis($[\mathbf{v}_0, \ldots, \mathbf{v}_n]$):
  $[\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*]$ = orthogonalize($[\mathbf{v}_0, \ldots, \mathbf{v}_n]$)
  Return $[\mathbf{v}_i$ for i in $\{0, \ldots, n\}$ if $\mathbf{v}_i^*$ is not the zero vector$]$

```
def orthogonalize(vlist):
 vstarlist = []
 for v in vlist:
   vstarlist.append(
    project_orthogonal(v, vstarlist))
 return vstarlist
```

Example: orthogonalize($[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6]$) returns $[\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_3^*, \mathbf{v}_4^*, \mathbf{v}_5^*, \mathbf{v}_6^*]$

Suppose $\mathbf{v}_2^*$, $\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero vectors.

In iteration 3 iteration of `orthogonalize`, project_orthogonal($\mathbf{v}_3, [\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*]$) computes $\mathbf{v}_3^*$:

- subtract projection of $\mathbf{v}_3$ along $\mathbf{v}_0^*$,
- subtract projection along $\mathbf{v}_1^*$,
- subtract projection along $\mathbf{v}_2^*$—but since $\mathbf{v}_2^* = \mathbf{0}$, the projection is the zero vector

Result is the same as project_orthogonal($\mathbf{v}_3, [\mathbf{v}_0^*, \mathbf{v}_1^*]$). Zero starred vectors are ignored.
Thus orthogonalize($[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6]$) would return $[\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*]$.
Since $[\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*]$ is a basis for $\mathcal{V} = \text{Span}\,\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$

Correctness of find_subset_basis

def find_subset_basis($[\mathbf{v}_0, \ldots, \mathbf{v}_n]$):
  $[\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*] =$ `orthogonalize`$([\mathbf{v}_0, \ldots, \mathbf{v}_n])$
  Return $[\mathbf{v}_i$ for i in $\{0, \ldots, n\}$ if $\mathbf{v}_i^*$ is not the zero vector$]$

```
def orthogonalize(vlist):
 vstarlist = []
 for v in vlist:
   vstarlist.append(
    project_orthogonal(v, vstarlist))
 return vstarlist
```

Suppose $\mathbf{v}_2^*$, $\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero vectors.

In iteration 3 iteration of `orthogonalize`, project_orthogonal$(\mathbf{v}_3, [\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*])$ computes $\mathbf{v}_3^*$:

- subtract projection of $\mathbf{v}_3$ along $\mathbf{v}_0^*$,
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Result is the same as project_orthogonal$(\mathbf{v}_3, [\mathbf{v}_0^*, \mathbf{v}_1^*])$. Zero starred vectors are ignored.
Thus `orthogonalize`$([\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6])$ would return $[\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*]$.
Since $[\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*]$ is a basis for $\mathcal{V} = \text{Span}\ \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$
and $[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6]$ spans the same space, and has the same cardinality.
$[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6]$ is also a basis for $\mathcal{V}$.

Correctness of find_subset_basis

Another way to justify `find_subset_basis`...

Here's the matrix equation expressing original vectors in terms of starred vectors:

$$\left[\begin{array}{c|c|c|c|c} \\ \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ \\ \end{array}\right] = \left[\begin{array}{c|c|c|c|c} \\ \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \cdots & \mathbf{v}_n^* \\ \\ \end{array}\right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & & \alpha_{0n} \\ & 1 & \alpha_{12} & & \alpha_{1n} \\ & & 1 & & \alpha_{2n} \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

Correctness of find_subset_basis

$$\begin{bmatrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \mathbf{v}_3^* & \mathbf{v}_4^* & \mathbf{v}_5^* & \mathbf{v}_6^* \end{bmatrix} \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06} \\ & 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ & & 1 & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ & & & 1 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ & & & & 1 & \alpha_{45} & \alpha_{46} \\ & & & & & 1 & \alpha_{56} \\ & & & & & & 1 \end{bmatrix}$$

Let $\mathcal{V} = \text{Span}\ \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$.

Suppose $\mathbf{v}_2^*$, $\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero vectors.

Delete zero columns and the corresponding rows of the triangular matrix. Shows

$\text{Span}\ \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\} \subseteq \text{Span}\ \{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\}$

so $\{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\}$ is a basis for $\mathcal{V}$

Delete corresponding original columns $\mathbf{v}_2$, $\mathbf{v}_4$, $\mathbf{v}_5$.

Resulting triangular matrix is invertible. Move it to other side.

Shows $\text{Span}\ \{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_6^*\} \subseteq \text{Span}\ \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ so $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ is basis for $\mathcal{V}$. QED

Correctness of find_subset_basis

$$\begin{bmatrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_3 & \mathbf{v}_6 \end{bmatrix}$$

Let $\mathcal{V} = \text{Span } \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$.

Suppose $\mathbf{v}_2^*$, $\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero vectors.

$$= \begin{bmatrix} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_3^* & \mathbf{v}_6^* \end{bmatrix} \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06} \\ & 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ & & & 1 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ & & & & & & 1 \end{bmatrix}$$

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so $\{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\}$ is a basis for $\mathcal{V}$

Delete corresponding original columns $\mathbf{v}_2$, $\mathbf{v}_4$, $\mathbf{v}_5$.

Resulting triangular matrix is invertible. Move it to other side.

Shows $\text{Span } \{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_6^*\} \subseteq \text{Span } \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ so $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ is basis for $\mathcal{V}$. QED

Correctness of find_subset_basis

$$\left[\begin{array}{c|c|c|c} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_3 & \mathbf{v}_6 \end{array}\right]$$

Let $\mathcal{V} = \text{Span}\,\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$.

Suppose $\mathbf{v}_2^*$, $\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero vectors.

$$= \left[\begin{array}{c|c|c|c} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_3^* & \mathbf{v}_6^* \end{array}\right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06} \\ & 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ & & & 1 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ & & & & & & 1 \end{bmatrix}$$

Delete zero columns and the corresponding rows of the triangular matrix. Shows
$\text{Span}\,\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\} \subseteq \text{Span}\,\{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\}$
so $\{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\}$ is a basis for $\mathcal{V}$

Delete corresponding original columns $\mathbf{v}_2$, $\mathbf{v}_4$, $\mathbf{v}_5$.

Resulting triangular matrix is invertible. Move it to other side.

Shows $\text{Span}\,\{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_6^*\} \subseteq \text{Span}\,\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ so $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ is basis for $\mathcal{V}$. QED

Correctness of find_subset_basis

$$\begin{bmatrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_3 & \mathbf{v}_6 \end{bmatrix}$$

Let $\mathcal{V} = \text{Span } \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$.

Suppose $\mathbf{v}_2^*$, $\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero vectors.

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Delete zero columns and the corresponding rows of the triangular matrix. Shows
$\text{Span } \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\} \subseteq \text{Span } \{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\}$
so $\{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\}$ is a basis for $\mathcal{V}$
Delete corresponding original columns $\mathbf{v}_2$, $\mathbf{v}_4$, $\mathbf{v}_5$.
Resulting triangular matrix is invertible. Move it to other side.
Shows $\text{Span } \{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_6^*\} \subseteq \text{Span } \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ so $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ is basis for $\mathcal{V}$. QED

Correctness of find_subset_basis

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$$= \begin{bmatrix} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_3^* & \mathbf{v}_6^* \end{bmatrix} \begin{bmatrix} 1 & \alpha_{01} & \alpha_{03} & \alpha_{06} \\ & 1 & \alpha_{13} & \alpha_{16} \\ & & 1 & \alpha_{36} \\ & & & 1 \end{bmatrix}$$

Delete zero columns and the corresponding rows of the triangular matrix. Shows
Span $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\} \subseteq$ Span $\{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\}$
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Delete corresponding original columns $\mathbf{v}_2$, $\mathbf{v}_4$, $\mathbf{v}_5$.
Resulting triangular matrix is invertible. Move it to other side.
Shows Span $\{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_6^*\} \subseteq$ Span $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ so $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ is basis for $\mathcal{V}$. QED

Correctness of find_subset_basis

$$\begin{bmatrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_3 & \mathbf{v}_6 \end{bmatrix}$$

Let $\mathcal{V} = \text{Span}\ \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$.

Suppose $\mathbf{v}_2^*$, $\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero vectors.

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Delete zero columns and the corresponding rows of the triangular matrix. Shows
Span $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\} \subseteq$ Span $\{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\}$
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Delete corresponding original columns $\mathbf{v}_2$, $\mathbf{v}_4$, $\mathbf{v}_5$.
Resulting triangular matrix is invertible. Move it to other side.
Shows Span $\{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_6^*\} \subseteq$ Span $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ so $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ is basis for $\mathcal{V}$. QED

Correctness of find_subset_basis

$$\begin{bmatrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_3 & \mathbf{v}_6 \end{bmatrix} \begin{bmatrix} 1 & \alpha_{01} & \alpha_{03} & \alpha_{06} \\ & 1 & \alpha_{13} & \alpha_{16} \\ & & 1 & \alpha_{36} \\ & & & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_3^* & \mathbf{v}_6^* \end{bmatrix}$$

Let $\mathcal{V} = \text{Span}\ \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$.

Suppose $\mathbf{v}_2^*$, $\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero vectors.

Delete zero columns and the corresponding rows of the triangular matrix. Shows
$\text{Span}\ \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\} \subseteq \text{Span}\ \{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\}$
so $\{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\}$ is a basis for $\mathcal{V}$
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Shows $\text{Span}\ \{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_6^*\} \subseteq \text{Span}\ \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ so $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ is basis for $\mathcal{V}$. QED

Roundoff error in computing a basis

In principle the following algorithm computes a basis for Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$:

def find_basis($[\mathbf{v}_1, \ldots, \mathbf{v}_n]$)
    Use `orthogonalize` to compute $[\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*]$
    Return the list consisting of the nonzero vectors in this list.

However: the computer uses floating-point calculations.

Due to round-off error, the vectors that are supposed to be zero won't be exactly zero.

Instead, consider a vector $\mathbf{v}$ to be zero if $\mathbf{v} * \mathbf{v}$ is very small (e.g. smaller than 10^{-20}):

def find_basis($[\mathbf{v}_1, \ldots, \mathbf{v}_n]$)
    Use `orthogonalize` to compute $[\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*]$
    Return the list consisting of vectors in this list
    that are nearly zero vectors

Can use this procedure in turn to define `rank(vlist)` and `is_independent(vlist)`.

Algorithm for finding basis for null space

Now let's find null space of matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Here's the matrix equation expressing original vectors in terms of starred vectors:

$$\left[\begin{array}{c|c|c|c|c} \\ \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ \\ \end{array}\right] = \left[\begin{array}{c|c|c|c|c} \\ \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \cdots & \mathbf{v}_n^* \\ \\ \end{array}\right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & & \alpha_{0n} \\ & 1 & \alpha_{12} & & \alpha_{1n} \\ & & 1 & & \alpha_{2n} \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

Can transform this to express starred vectors in terms of original vectors.

$$\left[\begin{array}{c|c|c|c|c} \\ \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ \\ \end{array}\right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & & \alpha_{0n} \\ & 1 & \alpha_{12} & & \alpha_{1n} \\ & & 1 & & \alpha_{2n} \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}^{-1} = \left[\begin{array}{c|c|c|c|c} \\ \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \cdots & \mathbf{v}_n^* \\ \\ \end{array}\right]$$

Basis for null space

$$\left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array}\right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06} \\ & 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ & & 1 & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ & & & 1 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ & & & & 1 & \alpha_{45} & \alpha_{46} \\ & & & & & 1 & \alpha_{56} \\ & & & & & & 1 \end{bmatrix}^{-1}$$

$$= \left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \mathbf{v}_3^* & \mathbf{v}_4^* & \mathbf{v}_5^* & \mathbf{v}_6^* \end{array}\right]$$

Suppose $\mathbf{v}_2^*$, $\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are (approximately) zero vectors.

- Corresp. cols of inverse triang. matrix are vecs of null space of leftmost matrix.
- These columns are clearly linearly independent so they span a basis of dimension 3.
- Rank-Nullity Theorem shows that the null space has dimension 3 so these columns are a basis for null space.

Computing basis for null space

Write $\left[\begin{array}{c|c|c} \mathbf{v}_0 & \cdots & \mathbf{v}_n \end{array}\right] \left[\begin{array}{c} T \end{array}\right]^{-1} = \left[\begin{array}{c|c|c} \mathbf{v}_0^* & \cdots & \mathbf{v}_n^* \end{array}\right]$

Then one basis for null space consists of columns of T^{-1} corresponding to zero vectors among $\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*$.

How to compute columns of inverse of T?

Use matrix-matrix equation $\left[\begin{array}{c} T \end{array}\right] T^{-1} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$ Column i of T^{-1} is a solution to matrix-vector equation $T\mathbf{x} = \mathbf{e}_i$, where $\mathbf{e}_i$ is standard basis vector i

$$\mathbf{e}_i = [0, \ldots, 0, \underbrace{1}_{\text{entry } i}, 0, \ldots, 0]$$

How to solve matrix-vector equation? backward substitution

Orthogonal complement

Let $\mathcal{U}$ be a subspace of $\mathcal{W}$.
For each vector $\mathbf{b}$ in $\mathcal{W}$, we can write $\mathbf{b} = \mathbf{b}^{||\mathcal{U}} + \mathbf{b}^{\perp\mathcal{U}}$ where

- $\mathbf{b}^{||\mathcal{U}}$ is in $\mathcal{U}$, and
- $\mathbf{b}^{\perp\mathcal{U}}$ is orthogonal to every vector in $\mathcal{U}$.

Let $\mathcal{V}$ be the set $\{\mathbf{b}^{\perp\mathcal{U}} \ : \ \mathbf{b} \in \mathcal{W}\}$.
Definition: We call $\mathcal{V}$ the *orthogonal complement* of $\mathcal{U}$ in $\mathcal{W}$

Easy observations:

- Every vector in $\mathcal{V}$ is orthogonal to every vector in $\mathcal{U}$.
- Every vector $\mathbf{b}$ in $\mathcal{W}$ can be written as the sum of a vector in $\mathcal{U}$ and a vector in $\mathcal{V}$.

Maybe $\mathcal{U} \oplus \mathcal{V} = \mathcal{W}$? To show direct sum of $\mathcal{U}$ and $\mathcal{V}$ is defined, we need to show that the only in vector that is in both $\mathcal{U}$ and $\mathcal{V}$ is the zero vector.

Any vector $\mathbf{w}$ in both $\mathcal{U}$ and $\mathcal{V}$ is orthogonal to itself.
Thus $0 = \langle \mathbf{w}, \mathbf{w} \rangle = \|\mathbf{w}\|^2$.
By Property N2 of norms, that means $\mathbf{w} = \mathbf{0}$.

Therefore $\mathcal{U} \oplus \mathcal{V} = \mathcal{W}$. **Recall:** $\dim \mathcal{U} + \dim \mathcal{V} = \dim \mathcal{U} \oplus \mathcal{V}$

Orthogonal complement: example

Example: Let $\mathcal{U} = \text{Span } \{[1,1,0,0],[0,0,1,1]\}$. Let $\mathcal{V}$ denote the orthogonal complement of $\mathcal{U}$ in $\mathbb{R}^4$. What vectors form a basis for $\mathcal{V}$?

Every vector in $\mathcal{U}$ has the form $[a, a, b, b]$.

Therefore any vector of the form $[c, -c, d, -d]$ is orthogonal to every vector in $\mathcal{U}$.

Every vector in $\text{Span } \{[1,-1,0,0],[0,0,1,-1]\}$ is orthogonal to every vector in $\mathcal{U}$....
... so $\text{Span } \{[1,-1,0,0],[0,0,1,-1]\}$ is a subspace of $\mathcal{V}$, the orthogonal complement of $\mathcal{U}$ in $\mathbb{R}^4$.

Is it the whole thing?

$\mathcal{U} \oplus \mathcal{V} = \mathbb{R}^4$ so $\dim \mathcal{U} + \dim \mathcal{V} = 4$.

$\{[1,1,0,0],[0,0,1,1]\}$ is linearly independent so $\dim \mathcal{U} = 2$... so $\dim \mathcal{V} = 2$

$\{[1,-1,0,0],[0,0,1,-1]\}$ is linearly independent
so $\dim \text{Span } \{[1,-1,0,0],[0,0,1,-1]\}$ is also 2....
so $\text{Span } \{[1,-1,0,0],[0,0,1,-1]\} = \mathcal{V}$.

Orthogonal complement: example

Example: Find a basis for the null space of $A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ \hline 0 & 5 & 1 & 2 \\ \hline 0 & 2 & 5 & 6 \end{bmatrix}$

By the dot-product definition of matrix-vector multiplication, a vector $\mathbf{v}$ is in the null space of A if the dot-product of each row of A with $\mathbf{v}$ is zero.

Thus the null space of A equals the orthogonal complement of Row A in $\mathbb{R}^4$.

Since the three rows of A are linearly independent, we know $\dim \text{Row } A = 3$...

so the dimension of the orthogonal complement of Row A in $\mathbb{R}^4$ is $4 - 3 = 1$....

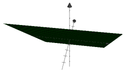
The vector $[1, \frac{1}{10}, \frac{13}{20}, \frac{-23}{40}]$ has a dot-product of zero with every row of A...

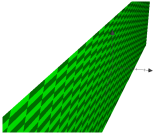
so this vector forms a basis for the orthogonal complement.

and thus a basis for the null space of A.

Using orthogonalization to find intersection of geometric objects

Example: Find the intersection of





- the plane spanned by $[1, 2, -2]$ and $[0, 1, 1]$
- the plane spanned by $[1, 0, 0]$ and $[0, 1, -1]$

The orthogonal complement in $\mathbb{R}^3$ of the first plane is Span $\{[4, -1, 1]\}$.
Therefore first plane is $\{[x, y, z] \in \mathbb{R}^3 \ : \ [4, -1, 1] \cdot [x, y, z] = 0\}$
The orthogonal complement in $\mathbb{R}^3$ of the second plane is Span $\{[0, 1, 1]\}$.
Therefore second plane is $\{[x, y, z] \in \mathbb{R}^3 \ : \ [0, 1, 1] \cdot [x, y, z] = 0\}$

The intersection of these two sets is the set

$$\{[x, y, z] \in \mathbb{R}^3 \ : \ [4, -1, 1] \cdot [x, y, z] = 0 \text{ and } [0, 1, 1] \cdot [x, y, z] = 0\}$$

How to find a basis for this solution set? We saw (in Lecture of 10/30) that is is just a basis for the null space of $A = \left[\begin{array}{ccc} 4 & -1 & 1 \\ \hline 0 & 1 & 1 \end{array}\right]$

The null space of A is the orthogonal complement of Span $\{[4, -1, 1], [0, 1, 1]\}$ in $\mathbb{R}^3$...
which is Span $\{[1, 2, -2]\}$