Quiz

Prove one of the following two statements:

1. If M is invertible then Row $MA = $ Row A (assumes it is legal to multiply MA)
2. The union of a basis for subspace $\mathcal{U}$ and subspace $\mathcal{V}$ is a basis for $\mathcal{U} \oplus \mathcal{V}$ (assumes it is legal to take the direct sum $\mathcal{U} \oplus \mathcal{V}$)

Quiz

What are the rank and nullity of the following matrices?

- $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
- $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Using Gaussian elimination for other problems

So far:

- we know how to use Gaussian elimination to transform a matrix into echelon form;
- nonzero rows form a basis for row space of original matrix

We can do other things with Gaussian elimination:

- Solve linear systems (used in e.g. *Lights Out*)
- Find vectors in null space (used in e.g. integer factoring)

Key idea: keep track of transformations performed in putting matrix in echelon form.

Gaussian elimination: recording the transformations

- Maintain M (initially identity) and U (initially A)
- Whatever transformations you do to U, do same transformations to M

	0	1	2	3
0	1	0	0	0
1	0	1	0	0
2	0	0	1	0
3	0	0	0	1

$*$

	A	B	C	D	
0	0	0	1	1	
1	1	0	1	1	✓
2	1	0	0	1	
3	1	1	1	1	

$=$

	A	B	C	D
0	0	0	1	1
1	1	0	1	1
2	1	0	0	1
3	1	1	1	1

ColumnA:
select row 1
add it to rows 2,3

	0	1	2	3
0	1	0	0	0
1	0	1	0	0
2	0	1	1	0
3	0	1	0	1

$*$

	A	B	C	D	
0	0	0	1	1	✓
1	1	0	1	1	✓
2	1	0	0	1	
3	1	1	1	1	✓

$=$

	A	B	C	D
0	0	0	1	1
1	1	0	1	1
2	0	0	1	0
3	0	1	0	0

ColumnB:
select row 3
add it to no rows

ColumnC:
select row 0
add it to row 2

	0	1	2	3
0	1	0	0	0
1	0	1	0	0
2	1	1	1	0

$*$

	A	B	C	D	
0	0	0	1	1	✓
1	1	0	1	1	✓
2	1	0	0	1	✓

$=$

	A	B	C	D
0	0	0	1	1
1	1	0	1	1
2	0	0	0	1

ColumnD:
select row 2
done

Code for finding transformation to echelon form

- Initialize `rowlist` to be list of rows of A
- Initialize `M_rowlist` to be list of rows of identity matrix

```
for c in sorted(col_labels, key=str):
    rows_with_nonzero = [r for r in rows_left if rowlist[r][c] != 0]
    if rows_with_nonzero != []:
        pivot = rows_with_nonzero[0]
        rows_left.remove(pivot)
        new_M_rowlist.append(M_rowlist[pivot])
        for r in rows_with_nonzero[1:]:
            multiplier = rowlist[r][c]/rowlist[pivot][c]
            rowlist[r] -= multiplier*rowlist[pivot]
            M_rowlist[r] -= multiplier*M_rowlist[pivot]

    for r in rows_left: new_M_rowlist.append(M_rowlist[r])
```

Finally, return matrix M formed from `new_M_rowlist`

Code provided in module `echelon`

Gaussian Elimination: Finding basis for null space

Instead of finding basis for null space of A, find basis for $\{\mathbf{u} \;:\; \mathbf{u} * A = \mathbf{0}\} = \text{Null } A^T$

Input:

	A	B	C	D
0	1	0	1	0
1	1	1	1	0
2	0	1	0	1
3	1	1	1	1
4	0	0	0	1

Find M such that the matrix $U = MA$ is in echelon form and M is invertible

M:

	0	1	2	3	4
0	1	0	0	0	0
1	1	1	0	0	0
2	1	1	1	0	0
3	1	0	1	1	0
4	1	1	1	0	1

$*$

A:

	A	B	C	D
0	1	0	1	0
1	1	1	1	0
2	0	1	0	1
3	1	1	1	1
4	0	0	0	1

$=$

U:

	0	1	2	3
0	1	0	1	0
1	0	1	0	0
2	0	0	0	1
3	0	0	0	0
4	0	0	0	0

Last two rows of U are zero vectors.

- Row 3 of U is (row 3 of M) * A

Gaussian Elimination: Finding basis for null space

Find M such that the matrix $U = MA$ is in echelon form and M is invertible

M:

	0	1	2	3	4
0	1	0	0	0	0
1	1	1	0	0	0
2	1	1	1	0	0
3	1	0	1	1	0
4	1	1	1	0	1

$*$

A:

	A	B	C	D
0	1	0	1	0
1	1	1	1	0
2	0	1	0	1
3	1	1	1	1
4	0	0	0	1

$=$

U:

	0	1	2	3
0	1	0	1	0
1	0	1	0	0
2	0	0	0	1
3	0	0	0	0
4	0	0	0	0

Last two rows of U are zero vectors.

- Row 3 of U is (row 3 of M) * A
- Row 4 of U is (row 4 of M) * A

Therefore two rows in $\{\mathbf{u} : \mathbf{u} * A = \mathbf{0}\}$ are rows 3 and 4 of M.
To show that these two rows form a basis for $\{\mathbf{u} : \mathbf{u} * A = \mathbf{0}\}$....
$\dim \text{Row } A = 3$. By Rank-Nullity Theorem, $\dim \text{Row } A + \dim \text{Null } A^T = \text{number of rows} = 5$.
Shows that $\dim \text{Null } A^T = 2$. Since M is invertible, all its rows are linearly independent.

Gaussian Elimination: Solving system of equations

Key idea: keep track of transformations performed in putting matrix in echelon form.

Given matrix A, compute matrices M and U such that $MA = U$

- U is in echelon form
- M is invertible

To solve $A\mathbf{x} = \mathbf{b}$:

- Compute M and U so that $MA = U$
- Compute the matrix-vector product $M\mathbf{b}$, and solve $U\mathbf{x} = M\mathbf{b}$.

Claim: This gives correct solution to $A\mathbf{x} = \mathbf{b}$

Proof: Suppose $\mathbf{v}$ is a solution to $U\mathbf{x} = M\mathbf{b}$, so $U\mathbf{v} = M\mathbf{b}$

- Multiply both sides by M^{-1}: $M^{-1}(U\mathbf{v}) = M^{-1}(M\mathbf{b})$
- Use associativity: $(M^{-1}U)\mathbf{v} = (M^{-1}M)\mathbf{b}$
- Cancel M^{-1} and M: $(M^{-1}U)\mathbf{v} = \mathbb{1}\mathbf{b}$
- Use $M^{-1}U = A$: $A\mathbf{v} = \mathbb{1}\mathbf{b} = \mathbf{b}$

How to solve $U\mathbf{x} = M\mathbf{b}$?

- If U is triangular, can solve using *back-substitution* (`triangular_solve`)
- In general, can use similar algorithm