Matrix invertibility

Rank-Nullity Theorem: For any n-column matrix A,

$$\text{nullity } A + \text{rank } A = n$$

Corollary: Let A be an $R \times C$ matrix. Then A is invertible if and only if $|R| = |C|$ and the columns of A are linearly independent.

Proof: Let $\mathbb{F}$ be the field. Define $f : \mathbb{F}^C \longrightarrow \mathbb{F}^R$ by $f(\mathbf{x}) = A\mathbf{x}$.
Then A is an invertible matrix if and only if f is an invertible function.

$$\begin{aligned}\text{The function } f \text{ is invertible} \quad & \text{iff} \quad \dim \operatorname{Ker} f = 0 \text{ and } \dim \mathbb{F}^C = \dim \mathbb{F}^R \\ & \text{iff} \quad \text{nullity } A = 0 \text{ and } |C| = |R|.\end{aligned}$$

$$\begin{aligned}\text{nullity } A = 0 \quad & \text{iff} \quad \dim \operatorname{Null} A = 0 \\ & \text{iff} \quad \operatorname{Null} A = \{\mathbf{0}\} \\ & \text{iff} \quad \text{the only vector } \mathbf{x} \text{ such that } A\mathbf{x} = \mathbf{0} \text{ is } \mathbf{x} = \mathbf{0} \\ & \text{iff} \quad \text{the columns of } A \text{ are linearly independent.}\end{aligned}$$

QED

Matrix invertibility examples

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is not square so cannot be invertible.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is square and its columns are linearly independent so it is invertible.

$\left[\begin{array}{c|c|c} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{array}\right]$ is square but columns not linearly independent so it is not invertible.

Transpose of invertible matrix is invertible

Theorem: The transpose of an invertible matrix is invertible.

$$A = \left[\begin{array}{c|c|c} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{array}\right] = \left[\begin{array}{c} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_n \end{array}\right] \qquad A^T = \left[\begin{array}{c|c|c} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{array}\right]$$

Proof: Suppose A is invertible. Then A is square and its columns are linearly independent. Let n be the number of columns. Then rank $A = n$.

Because A is square, it has n rows. By Rank Theorem, rows are linearly independent.

Columns of transpose A^T are rows of A, so columns of A^T are linearly independent.

Since A^T is square and columns are linearly independent, A^T is invertible. QED

More matrix invertibility

Earlier we proved: *If A has an inverse A^{-1} then AA^{-1} is identity matrix*

Converse: If BA is identity matrix then A and B are inverses? **Not always true.**

Theorem: *Suppose A and B are square matrices such that BA is an identity matrix $\mathbb{1}$. Then A and B are inverses of each other.*

Proof: To show that A is invertible, need to show its columns are linearly independent.

Let $\mathbf{u}$ be any vector such that $A\mathbf{u} = \mathbf{0}$. Then $B(A\mathbf{u}) = B\mathbf{0} = \mathbf{0}$.
On the other hand, $(BA)\mathbf{u} = \mathbb{1}\mathbf{u} = \mathbf{u}$, so $\mathbf{u} = \mathbf{0}$.

This shows A has an inverse A^{-1}. Now must show $B = A^{-1}$. We know $AA^{-1} = \mathbb{1}$.

$$\begin{aligned}
BA &= \mathbb{1} \\
(BA)A^{-1} &= \mathbb{1}A^{-1} && \text{by multiplying on the right by } A^{-1} \\
(BA)A^{-1} &= A^{-1} \\
B(AA^{-1}) &= A^{-1} && \text{by associativity of matrix-matrix mult} \\
B\,\mathbb{1} &= A^{-1} \\
B &= A^{-1}
\end{aligned}$$

QED

Representations of vector spaces

Two important ways to represent a vector space:

As the solution set of homogeneous linear system

$$\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$$

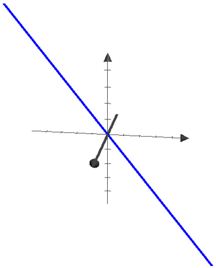
Equivalently, Null $\begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$

As Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$

Equivalently,

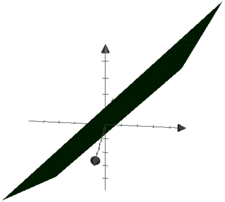
Col $\left[\begin{array}{c|c|c} & & \\ \mathbf{b}_1 & \cdots & \mathbf{b}_k \\ & & \end{array}\right]$

Conversions between the two representations



$\{[x, y, z] \;:\; [4, -1, 1] \cdot [x, y, z] = 0, \; [0, 1, 1] \cdot [x, y, z] = 0\}$

Span $\{[1, 2, -2]\}$



Span $\{[4, -1, 1], [0, 1, 1]\}$

$\{[x, y, z] \;:\; [1, 2, -2] \cdot [x, y, z] = 0\}$

Conversions for affine spaces?

- From representation as solution set of linear system to representation as affine hull
- From representation as affine hull to representation as solution set of linear system

Conversions for affine spaces?

From representation as solution set of linear system to representation as affine hull

- *input:* linear system $A\mathbf{x} = \mathbf{b}$
- *output:* vectors whose affine hull is the solution set of the linear system.

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

1 Let $\mathbf{u}$ be one solution to the linear system.

$\mathbf{u} = [-0.5, 0.75, 0]$

2 Consider the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Its solution set, the null space of A, is a vector space $\mathcal{V}$.

3 Let $\mathbf{b}_1, \ldots, \mathbf{b}_k$ be generators for $\mathcal{V}$.

$\mathbf{b}_1 = [2, -3, 4]$

4 Then the solution set of the original linear system is the affine hull of $\mathbf{u}, \mathbf{b}_1 + \mathbf{u}, \mathbf{b}_2 + \mathbf{u}, \ldots, \mathbf{b}_k + \mathbf{u}$.

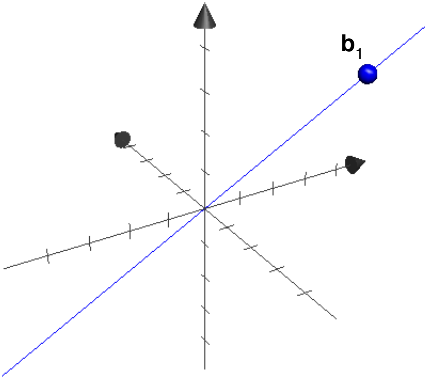
$[-0.5, -.75, 0]$ and
$[-0.5, -.75, 0] + [2, -3, 4]$

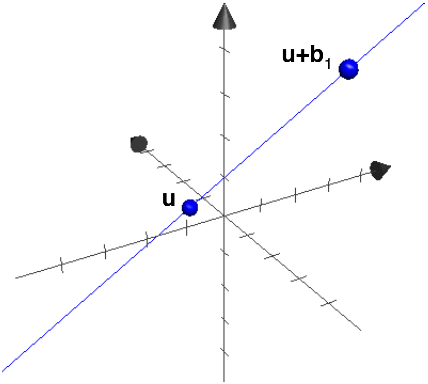
From representation as solution set to representation as affine hull

One solution to equation $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is $\mathbf{u} = [-0.5, 0.75, 0]$

Null space of $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ is Span $\{\mathbf{b}_1\}$:

Solution set of equation is $\mathbf{u} + \text{Span } \{\mathbf{b}_1\}$, i.e. the affine hull of $\mathbf{u}$ and $\mathbf{u} + \mathbf{b}_1$





Representations of vector spaces

Two important ways to represent a vector space:

As the solution set of homogeneous linear system

$$\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$$

Equivalently, Null $\begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$

As Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$

Equivalently,

Col $\left[\begin{array}{c|c|c} \mathbf{b}_1 & \cdots & \mathbf{b}_k \end{array}\right]$

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How to transform between these two representations?

Problem 1 (From left to right):

- *input:* homogeneous linear system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$,
- *output:* basis $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

Problem 2 (From right to left):

- *input:* independent vectors $\mathbf{b}_1, \ldots, \mathbf{b}_k$,
- *output:* homogeneous linear system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$ whose solution set equals

Reformulating Problem 1

- *input:* homogeneous linear system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0,$
- *output:* basis $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

Let's express this in the language of matrices:

- *input:* matrix $A = \begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$
- *output:* matrix $B = \left[\begin{array}{c|c|c} \mathbf{b}_1 & \cdots & \mathbf{b}_k \end{array}\right]$ such that Col $B =$ Null A

Can require the rows of the input matrix A to be linearly independent. (Discarding a superfluous row does not change the null space of A.)

Reformulating Problem 1

- *input:* homogeneous linear system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$,
- *output:* basis $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

Let's express this in the language of matrices:

- *input:* matrix $A = \begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$
- *output:* matrix $B = \left[\begin{array}{c|c|c} \mathbf{b}_1 & \cdots & \mathbf{b}_k \end{array}\right]$ with independent columns such that Col B = Null A

Can require the rows of the input matrix A to be linearly independent. (Discarding a superfluous row does not change the null space of A.)

Reformulating Problem 1

- *input:* homogeneous linear system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$,
- *output:* basis $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

Let's express this in the language of matrices:

- *input:* matrix $A = \begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$ with independent rows
- *output:* matrix $B = \left[\begin{array}{c|c|c} \mathbf{b}_1 & \cdots & \mathbf{b}_k \end{array}\right]$ with independent columns such that Col B = Null A

Can require the rows of the input matrix A to be linearly independent. (Discarding a superfluous row does not change the null space of A.)

Reformulating the reformulation of Problem 1

- *input:* matrix A with independent rows
- *output:* matrix B with independent columns such that Col B = Null A

By Rank-Nullity Theorem, rank A + nullity $A = n$

Because rows of A are linearly independent, rank $A = m$,

so $\boxed{m + \text{nullity } A = n}$

Requiring Col B = Null A is the same as requiring

(i) Col B is a subspace of Null A

(ii) $\dim \text{Col } B = \text{nullity } A$

Reformulating the reformulation of Problem 1

- *input:* matrix A with independent rows
- *output:* matrix B with independent columns such that $\text{Col } B = \text{Null } A$

By Rank-Nullity Theorem, $\text{rank } A + \text{nullity } A = n$
Because rows of A are linearly independent, $\text{rank } A = m$,
so $\boxed{m + \text{nullity } A = n}$

Requiring $\text{Col } B = \text{Null } A$ is the same as requiring

(i) $\text{Col } B$ is a subspace of $\text{Null } A \Longrightarrow$ same as requiring $AB = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$

(ii) $\dim \text{Col } B = \text{nullity } A$

Reformulating the reformulation of Problem 1

- *input:* matrix A with independent rows
- *output:* matrix B with independent columns such that Col $B =$ Null A

By Rank-Nullity Theorem, rank $A +$ nullity $A = n$
Because rows of A are linearly independent, rank $A = m$,
so $\boxed{m + \text{nullity } A = n}$

Requiring Col $B =$ Null A is the same as requiring

(i) Col B is a subspace of Null A $\Longrightarrow$ same as requiring $AB = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$

(ii) dim Col $B =$ nullity A $\Longrightarrow$ same as requiring number of columns of $B =$ nullity A
same as requiring number of columns of $B = n - m$

Reformulating the reformulation of Problem 1

- *input:* matrix A with independent rows
- *output:* matrix B with independent columns such that Col B = Null A

By Rank-Nullity Theorem, rank A + nullity $A = n$
Because rows of A are linearly independent, rank $A = m$,
so $\boxed{m + \text{nullity } A = n}$

Requiring Col B = Null A is the same as requiring

(i) Col B is a subspace of Null A $\Longrightarrow$ same as requiring $AB = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$

(ii) dim Col B = nullity A $\Longrightarrow$ same as requiring number of columns of B = nullity A
same as requiring number of columns of $B = n - m$

- *input:* $m \times n$ matrix A with independent rows
- *output:* matrix B with $n - m$ independent columns such that $AB = \begin{bmatrix} 0 \end{bmatrix}$

Hypothesize a procedure for reformulation of Problem 1

Problem 1:

- *input:* $m \times n$ matrix A with independent rows
- *output:* matrix B with $n - m$ independent columns such that $AB = \left[\begin{array}{c} 0 \end{array}\right]$

Define procedure `null_space_basis(M)` with this spec:

- *input:* $r \times n$ matrix M with independent rows
- *output:* matrix C with $n - r$ independent columns such that $MC = \left[\begin{array}{c} 0 \end{array}\right]$

Reformulating Problem 2

- *input:* independent vectors $\mathbf{b}_1, \ldots, \mathbf{b}_k$,
- *output:* homogeneous linear system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$ whose solution set equals Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$

Let's express this in the language of matrices:

- *input:* $n \times k$ matrix B with independent columns
- *output:* matrix A with independent rows such that Null $A =$ Col B

As before, Rank-Nullity Theorem implies

number of rows of $A +$ nullity $A =$ number of columns of A

As before, requiring Null $A =$ Col B is the same as requiring

(i) $AB = \left[\; \mathbf{0} \; \right]$

(ii) number of rows of $A = n - k$

- *input:* $n \times k$ matrix B with independent rows
- *output:* matrix A with $n - k$ independent rows such that $AB = \left[\; \mathbf{0} \; \right]$

Solving Problem 2 with the procedure for Problem 1

Problem 1:

- *input:* $m \times n$ matrix A with independent rows
- *output:* matrix B with $n - m$ independent columns such that $AB = \begin{bmatrix} 0 \end{bmatrix}$

Define procedure `null_space_basis(M)`

- *input:* $r \times n$ matrix M with independent rows
- *output:* matrix C with $n - r$ independent columns such that $MC = \begin{bmatrix} 0 \end{bmatrix}$

Problem 2:

- *input:* $n \times k$ matrix B with independent rows
- *output:* matrix A with $n - k$ independent rows such that $AB = \begin{bmatrix} 0 \end{bmatrix}$

To solve Problem 2, call `null_space_basis`(B^T).

Returns matrix A^T with independent columns such that $B^T A^T = \begin{bmatrix} 0 \end{bmatrix}$

Since B^T is $k \times n$ matrix, A^T has $n - k$ columns.

Therefore $AB = \begin{bmatrix} 0 \end{bmatrix}$ and A has $n - k$ independent rows. Therefore A is solution to Problem