Quiz

Suppose $\mathbf{u}_1, \dots, \mathbf{u}_n$ is a basis for $\mathcal{U}$ and $\mathbf{v}_1, \dots, \mathbf{v}_k$ is a basis for $\mathcal{V}$. Prove that $\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{v}_1, \dots, \mathbf{v}_k$ is a basis for $\mathcal{U} \oplus \mathcal{V}$.

Two parts to the proof:

1. Show that Span $\{\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{v}_1, \dots, \mathbf{v}_k\}$ is $\mathcal{U} \oplus \mathcal{V}$.
2. Show that $\{\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set.

Direct Sum

Let $\mathcal{U}$ and $\mathcal{V}$ be two vector spaces consisting of D-vectors over a field $\mathbb{F}$.

Definition: If $\mathcal{U}$ and $\mathcal{V}$ share only the zero vector then we define the *direct sum* of $\mathcal{U}$ and $\mathcal{V}$ to be the set

$$\{\mathbf{u}+\mathbf{v} \;: \mathbf{u} \in \mathcal{U}, \mathbf{v} \in \mathcal{V}\}$$

written $\mathcal{U} \oplus \mathcal{V}$

That is, $\mathcal{U} \oplus \mathcal{V}$ is the set of all sums of a vector in $\mathcal{U}$ and a vector in $\mathcal{V}$.

In Python, `[u+v for u in U for v in V]`

(But generally $\mathcal{U}$ and $\mathcal{V}$ are infinite so the Python is just suggestive.)

Direct Sum: Example

Vectors over $GF(2)$:

Example: Let $\mathcal{U} = \text{Span}\ \{1000, 0100\}$ and let $\mathcal{V} = \text{Span}\ \{0010\}$.

- Every nonzero vector in $\mathcal{U}$ has a one in the first or second position (or both) and nowhere else.
- Every nonzero vector in $\mathcal{V}$ has a one in the third position and nowhere else.

Therefore the only vector in both $\mathcal{U}$ and $\mathcal{V}$ is the zero vector.

Therefore $\mathcal{U} \oplus \mathcal{V}$ is defined.

$\mathcal{U} \oplus \mathcal{V} = \{0000 + 0000, 1000 + 0000, 0100 + 0000, 1100 + 0000, 0000 + 0010, 1000 + 0010, 0100 + 0010, 1100 + 0010\}$

which is equal to $\{0000, 1000, 0100, 1100, 0010, 1010, 0110, 1110\}$.

Direct Sum: Example

Vectors over $\mathbb{R}$:

Example: Let $\mathcal{U} = \text{Span } \{[1,2,1,2],[3,0,0,4]\}$ and let $\mathcal{V}$ be the null space of $\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$.

- The vector $[2,-2,-1,2]$ is in $\mathcal{U}$ because it is $[3,0,0,4]-[1,2,1,2]$
- It is also in $\mathcal{V}$ because

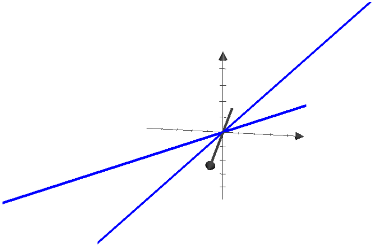
$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore we cannot form $\mathcal{U} \oplus \mathcal{V}$.

Direct Sum: Example

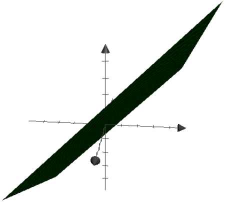
Vectors over $\mathbb{R}$:

Example:



- Let $\mathcal{U} = \text{Span}\ \{[4, -1, 1]\}$.
- Let $\mathcal{V} = \text{Span}\ \{[0, 1, 1]\}$.

The only intersection is at the origin, so $\mathcal{U} \oplus \mathcal{V}$ is defined.



- $\mathcal{U} \oplus \mathcal{V}$ is the set of vectors $\mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in \mathcal{U}$ and $\mathbf{v} \in \mathcal{V}$.
- This is just $\text{Span}\ \{[4, -1, 1], [0, 1, 1]\}$
- Plane containing the two lines

Properties of direct sum

Lemma: $\mathcal{U} \oplus \mathcal{V}$ is a vector space.

(Prove using Properties V1, V2, V3.)

Lemma: The union of

- a set of generators of $\mathcal{U}$, and
- a set of generators of $\mathcal{V}$

is a set of generators for $\mathcal{U} \oplus \mathcal{V}$.

Proof: Suppose $\mathcal{U} = \text{Span}\ \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and $\mathcal{V} = \text{Span}\ \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.
Then

- every vector in $\mathcal{U}$ can be written as $\alpha_1\, \mathbf{u}_1 + \cdots + \alpha_m\, \mathbf{u}_m$, and
- every vector in $\mathcal{V}$ can be written as $\beta_1\, \mathbf{v}_1 + \cdots + \beta_n\, \mathbf{v}_n$

so every vector in $\mathcal{U} \oplus \mathcal{V}$ can be written as

$$\alpha_1\, \mathbf{u}_1 + \cdots + \alpha_m\, \mathbf{u}_m \ + \ \beta_1\, \mathbf{v}_1 + \cdots + \beta_n\, \mathbf{v}_n$$

QED

Properties of direct sum

Direct Sum Basis Lemma:

Union of a basis of $\mathcal{U}$ and a basis of $\mathcal{V}$ is a basis of $\mathcal{U} \oplus \mathcal{V}$.

Proof: Clearly

- a basis of $\mathcal{U}$ is a set of generators for $\mathcal{U}$, and
- a basis of $\mathcal{V}$ is a set of generators for $\mathcal{V}$.

Therefore the previous lemma shows that

- the union of a basis for $\mathcal{U}$ and a basis for $\mathcal{V}$ is a generating set for $\mathcal{U} \oplus \mathcal{V}$.

We just need to show that the union is linearly independent.

Properties of direct sum

Direct Sum Basis Lemma:

Union of a basis of $\mathcal{U}$ and a basis of $\mathcal{V}$ is a basis of $\mathcal{U} \oplus \mathcal{V}$.

Proof, cont'd: Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be a basis for $\mathcal{U}$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for $\mathcal{V}$.
We need to show that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is independent.
Suppose

$$\mathbf{0} = \alpha_1 \, \mathbf{u}_1 + \cdots + \alpha_m \mathbf{u}_m + \beta_1 \, \mathbf{v}_1 + \cdots + \beta_n \, \mathbf{v}_n.$$

Then

$$\underbrace{\alpha_1 \, \mathbf{u}_1 + \cdots + \alpha_m \, \mathbf{u}_m}_{\text{in } \mathcal{U}} = \underbrace{(-\beta_1) \, \mathbf{v}_1 + \cdots + (-\beta_n) \, \mathbf{v}_n}_{\text{in } \mathcal{V}}$$

Left-hand side is a vector in $\mathcal{U}$, and right-hand side is a vector in $\mathcal{V}$.

By definition of $\mathcal{U} \oplus \mathcal{V}$, the only vector in both $\mathcal{U}$ and $\mathcal{V}$ is the zero vector.

This shows:

$$\mathbf{0} = \alpha_1 \, \mathbf{u}_1 + \cdots + \alpha_m \, \mathbf{u}_m$$

and

$$\mathbf{0} = (-\beta_1) \, \mathbf{v}_1 + \cdots + (-\beta_n) \, \mathbf{v}_n$$

Direct Sum

Direct-Sum Basis Lemma:
Union of a basis of $\mathcal{U}$ and a basis of $\mathcal{V}$ is a basis of $\mathcal{U} \oplus \mathcal{V}$.

Direct-Sum Dimension Corollary: $\dim \mathcal{U} + \dim \mathcal{V} = \dim \mathcal{U} \oplus \mathcal{V}$

Proof: A basis for $\mathcal{U}$ together with a basis for $\mathcal{V}$ forms a basis for $\mathcal{U} \oplus \mathcal{V}$. QED

Linear function invertibility

How to tell if a linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$ is invertible?

- *One-to-one?* f is one-to-one if its kernel is trivial. *Equivalent:* if its kernel has dimension zero.
- *Onto?* f is onto if its image equals its co-domain

Recall that the image of a function f with domain $\mathcal{V}$ is $\{f(\mathbf{v}) \ : \mathbf{v} \in \mathcal{V}\}$.

Lemma: The image of f is a subspace of $\mathcal{W}$.

How can we tell if the image of f equals $\mathcal{W}$?

Dimension Lemma: If $\mathcal{U}$ is a subspace of $\mathcal{W}$ then

Property D1: $\dim \mathcal{U} \leq \dim \mathcal{W}$, and

Property D2: if $\dim \mathcal{U} = \dim \mathcal{W}$ then $\mathcal{U} = \mathcal{W}$

Use Property D2 with $\mathcal{U} = \text{Im } f$.
Shows that the function f is onto iff $\dim \text{Im } f = \dim \mathcal{W}$

We conclude: $\boxed{f \text{ is invertible } \dim \text{Ker } f = 0 \text{ and } \dim \text{Im } f = \dim \mathcal{W}}$

Linear function invertibility

$$\boxed{f \text{ is one-to-one if } \dim \operatorname{Ker} f = 0 \text{ and } \dim \operatorname{Im} f = \dim \mathcal{W}}$$

How does this relate to dimension of the **domain**?

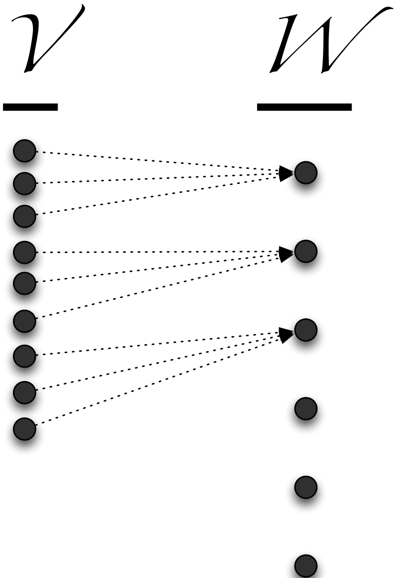
Conjecture For f to be invertible, need $\dim \mathcal{V} = \dim \mathcal{W}$.

Extracting an invertible function

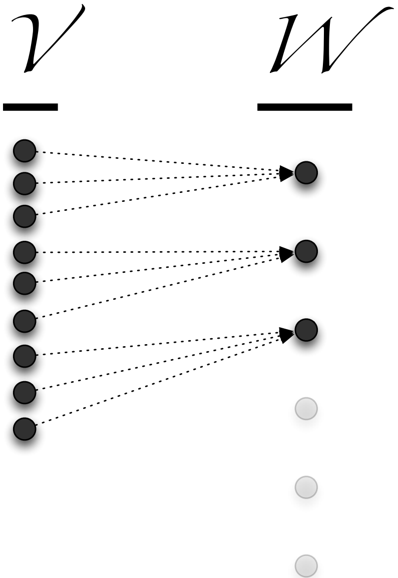
Starting with a linear function f we will extract a largest possible subfunction that is invertible.

Make it onto by setting co-domain to be image of f.

Make it one-to-one by getting rid of extra domain elements sharing same image.



Extracting an invertible function



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Make it onto by setting co-domain to be image of f.

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Extracting an invertible function

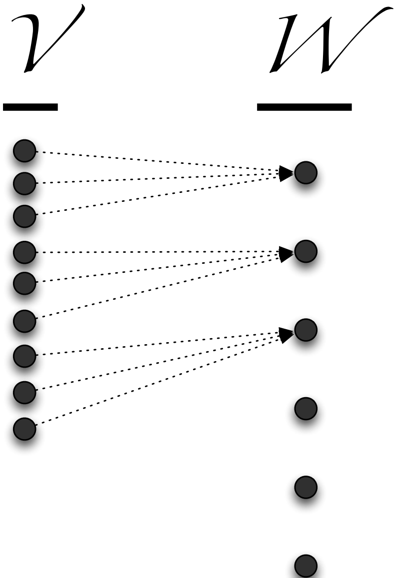
Start with linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

Step 1: Choose smaller co-domain $\mathcal{W}^*$

Step 2: Choose smaller domain $\mathcal{V}^*$

Step 3: Define function $f^* : \mathcal{V}^* \longrightarrow \mathcal{W}^*$ by $f^*(\mathbf{x}) = f(\mathbf{x})$

In fact, we will end up selecting a *basis* of $\mathcal{W}^*$ and a basis of $\mathcal{V}^*$.



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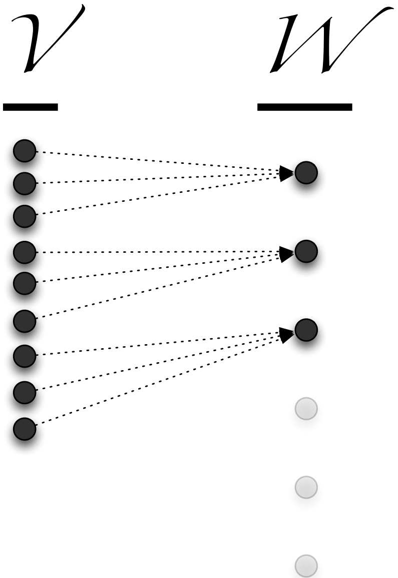

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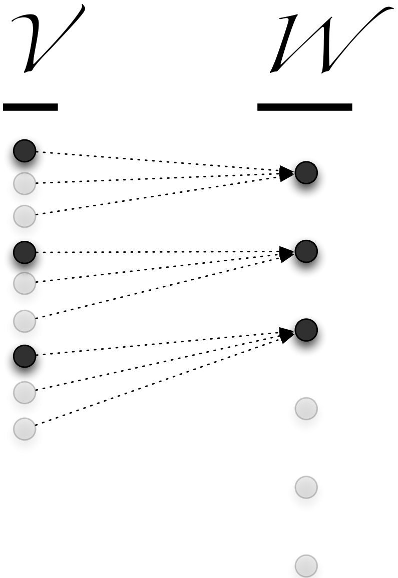


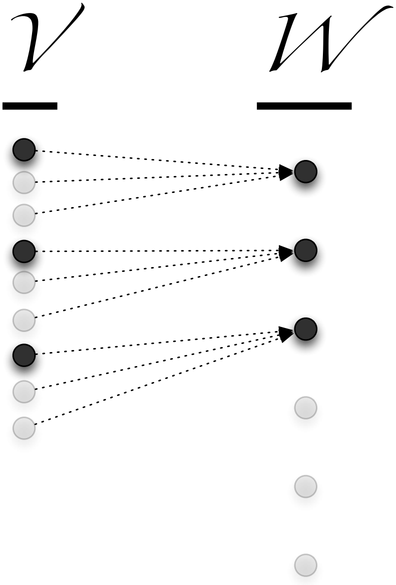
Extracting an invertible function from

- Choose smaller co-domain $\mathcal{W}^*$
  Let $\mathcal{W}^*$ be image of f

  Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis of $\mathcal{W}^*$

- Choose smaller domain $\mathcal{V}^*$
  Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be pre-images of $\mathbf{w}_1, \ldots, \mathbf{w}_r$
  That is, $f(\mathbf{v}_1) = \mathbf{w}_1, \ldots, f(\mathbf{v}_r) = \mathbf{w}_r$
  Let $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$

- Define function $f^* : \mathcal{V}^* \longrightarrow \mathcal{W}^*$
  by $f^*(\mathbf{x}) = f(\mathbf{x})$



We will show:

- f^* is onto
- f^* is one-to-one (kernel is trivial)
- Bonus: $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$

Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
  Let $\mathcal{W}^*$ be image of f

  Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis of $\mathcal{W}^*$

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Onto:

Let $\mathbf{w}$ be any vector in co-domain $\mathcal{W}^*$. There are scalars $\alpha_1, \ldots, \alpha_r$ such that

$$\mathbf{w} = \alpha_1 \mathbf{w}_1 + \cdots + \alpha_r \mathbf{w}_r$$

Because f is linear,

$$\begin{aligned} & f(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_r \mathbf{v}_r) \\ = \;& \alpha_1 f(\mathbf{v}_1) + \cdots + \alpha_r f(\mathbf{v}_r) \\ = \;& \alpha_1 \mathbf{w}_1 + \cdots + \alpha_r \mathbf{w}_r \end{aligned}$$

so $\mathbf{w}$ is image of $\alpha_1 \mathbf{v}_1 + \cdots + \alpha_r \mathbf{v}_r \in \mathcal{V}^*$.

QED

Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
  Let $\mathcal{W}^*$ be image of f

  Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis of $\mathcal{W}^*$
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  Let $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$
- Define function $f^* : \mathcal{V}^* \longrightarrow \mathcal{W}^*$
  by $f^*(\mathbf{x}) = f(\mathbf{x})$

We will show:

- f^* is onto
- f^* is one-to-one (kernel is trivial)
- Bonus: $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$

One-to-one:

By One-to-One Lemma, need only show kernel is trivial.

Suppose $\mathbf{v}^*$ is in $\mathcal{V}^*$ and $f(\mathbf{v}^*) = \mathbf{0}$

Because $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$, there are scalars $\alpha_1, \ldots, \alpha_r$ such that

$$\mathbf{v}^* = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_r \mathbf{v}_r$$

Applying f to both sides,

$$\begin{aligned} \mathbf{0} &= f(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_r \mathbf{v}_r) \\ &= \alpha_1 \mathbf{w}_1 + \cdots + \alpha_r \mathbf{w}_r \end{aligned}$$

Because $\mathbf{w}_1, \ldots, \mathbf{w}_r$ are linearly independent, $\alpha_1 = \cdots = \alpha_r = 0$

so $\mathbf{v}^* = \mathbf{0}$

QED

Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
  Let $\mathcal{W}^*$ be image of f

  Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis of $\mathcal{W}^*$
- Choose smaller domain $\mathcal{V}^*$
  Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be pre-images of $\mathbf{w}_1, \ldots, \mathbf{w}_r$
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- Define function $f^* : \mathcal{V}^* \longrightarrow \mathcal{W}^*$
  by $f^*(\mathbf{x}) = f(\mathbf{x})$

We will show:

- f^* is onto
- f^* is one-to-one (kernel is trivial)
- Bonus: $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$

Bonus: $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$

Need only show linear independence
Suppose $\mathbf{0} = \alpha_1\, \mathbf{v}_1 + \cdots + \alpha_r\, \mathbf{v}_r$

Applying f to both sides,

$$\begin{aligned} \mathbf{0} &= f(\alpha_1\, \mathbf{v}_1 + \cdots + \alpha_r\, \mathbf{v}_r) \\ &= \alpha_1\, \mathbf{w}_1 + \cdots + \alpha_r\, \mathbf{w}_r \end{aligned}$$

Because $\mathbf{w}_1, \ldots, \mathbf{w}_r$ are linearly independent,
$\alpha_1 = \cdots = \alpha_r = 0$. QED

Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
  Let $\mathcal{W}^*$ be image of f

  Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis of $\mathcal{W}^*$

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We will show:

- f^* is onto
- f^* is one-to-one (kernel is trivial)
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Example:

Let $A = \left[\begin{array}{c|c|c} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{array}\right]$, and define $\mathbf{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $f(\mathbf{x}) = A\mathbf{x}$.

Define $\mathcal{W}^* = \text{Im } f$
$= \text{Col } A = \text{Span } \{[1, 2, 1], [2, 1, 2], [1, 1, 1]\}$.

One basis for $\mathcal{W}^*$ is
$\mathbf{w}_1 = [0, 1, 0]$, $\mathbf{w}_2 = [1, 0, 1]$

Pre-images for $\mathbf{w}_1$ and $\mathbf{w}_2$:
$\mathbf{v}_1 = [\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]$ and $\mathbf{v}_2 = [-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$,
for then $A\mathbf{v}_1 = \mathbf{w}_1$ and $A\mathbf{v}_2 = \mathbf{w}_2$.

Let $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \mathbf{v}_2\}$

Then $f^* : \mathcal{V}^* \longrightarrow \text{Im } f$ is onto and one-to-one.

Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
  Let $\mathcal{W}^*$ be image of f

  Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis of $\mathcal{W}^*$
- Choose smaller domain $\mathcal{V}^*$
  Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be pre-images of $\mathbf{w}_1, \ldots, \mathbf{w}_r$
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We will show:

- f^* is onto
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To show about original function f:

original domain $\mathcal{V} = \text{Ker } f \oplus \mathcal{V}^*$
Must prove two things:

1. Ker f and $\mathcal{V}^*$ share only zero vector
2. every vector in $\mathcal{V}$ is the sum of a vector in Ker f and a vector in $\mathcal{V}^*$

We already showed kernel of f^* is trivial.
This shows only vector of Ker f in $\mathcal{V}^*$ is zero vector. —thing 1 is proved.

Let $\mathbf{v}$ be any vector in $\mathcal{V}$, and let $\mathbf{w} = f(\mathbf{v})$.
Since f^* is onto, its domain $\mathcal{V}^*$ contains a vector $\mathbf{v}^*$ such that $f(\mathbf{v}^*) = \mathbf{w}$
Therefore $f(\mathbf{v}) = f(\mathbf{v}^*)$ so $f(\mathbf{v}) - f(\mathbf{v}^*) = \mathbf{0}$
so $f(\mathbf{v} - \mathbf{v}^*) = \mathbf{0}$
Thus $\mathbf{u} = \mathbf{v} - \mathbf{v}^*$ is in Ker f
and $\mathbf{v} = \mathbf{u} + \mathbf{v}^*$ —thing 2 is proved.

Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
  Let $\mathcal{W}^*$ be image of f

  Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis of $\mathcal{W}^*$

- Choose smaller domain $\mathcal{V}^*$
  Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be pre-images of $\mathbf{w}_1, \ldots, \mathbf{w}_r$
  That is, $f(\mathbf{v}_1) = \mathbf{w}_1, \ldots, f(\mathbf{v}_r) = \mathbf{w}_r$
  Let $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$

- Define function $f^* : \mathcal{V}^* \longrightarrow \mathcal{W}^*$
  by $f^*(\mathbf{x}) = f(\mathbf{x})$

We will show:

- f^* is onto
- f^* is one-to-one (kernel is trivial)
- Bonus: $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$

original domain $\mathcal{V}$ = Ker $f \oplus \mathcal{V}^*$

Example: Let $A = \left[\begin{array}{c|c|c} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{array}\right]$, and define

$\mathbf{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $f(\mathbf{x}) = A\mathbf{x}$.

$\mathbf{v}_1 = [\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]$ and $\mathbf{v}_2 = [-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$

$\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \mathbf{v}_2\}$

Ker $f = \text{Span } \{[1, 1, -3]\}$

Therefore
$\mathcal{V} = (\text{Span } \{[1, 1, -3]\}) \oplus (\text{Span } \{\mathbf{v}_1, \mathbf{v}_2\})$

Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
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original domain $\mathcal{V} = \text{Ker } f \oplus \mathcal{V}^*$

By Direct-Sum Dimension Corollary,

$$\dim \mathcal{V} = \dim \text{Ker } f + \dim \mathcal{V}^*$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$,

$$\dim \mathcal{V}^* = r = \dim \text{Im } f$$

We have proved...

Kernel-Image Theorem:
For any linear function $f : \mathcal{V} \rightarrow \mathcal{W}$,

$$\dim \text{Ker } f + \dim \text{Im } f = \dim \mathcal{V}$$

Linear function invertibility, revisited

Kernel-Image Theorem:
For any linear function $f : \mathcal{V} \to \mathcal{W}$,

$$\dim \operatorname{Ker} f + \dim \operatorname{Im} f = \dim \mathcal{V}$$

Linear-Function Invertibility Theorem: Let $f : \mathcal{V} \longrightarrow \mathcal{W}$ be a linear function. Then f is invertible iff $\dim \operatorname{Ker} f = 0$ and $\dim \mathcal{V} = \dim \mathcal{W}$.

Proof: We saw before that f

- is one-to-one iff $\dim \operatorname{Ker} f = 0$
- is onto if $\dim \operatorname{Im} f = \dim \mathcal{W}$

Therefore f is invertible if $\dim \operatorname{Ker} f = 0$ and $\dim \operatorname{Im} f = \dim \mathcal{W}$.

Kernel-Image Theorem states $\dim \operatorname{Ker} f + \dim \operatorname{Im} f = \dim \mathcal{V}$

Therefore

$$\dim \operatorname{Ker} f = 0 \text{ and } \dim \operatorname{Im} f = \dim \mathcal{W}$$

$$\text{iff}$$

$$\dim \operatorname{Ker} f = 0 \text{ and } \dim \mathcal{V} = \dim \mathcal{W}$$

Rank-Nullity Theorem

Kernel-Image Theorem:
For any linear function $f : \mathcal{V} \to \mathcal{W}$,

$$\dim \text{Ker } f + \dim \text{Im } f = \dim \mathcal{V}$$

Apply Kernel-Image Theorem to the function $f(\mathbf{x}) = A\mathbf{x}$:

- $\text{Ker } f = \text{Null } A$
- $\dim \text{Im } f = \dim \text{Col } A = \text{rank } A$

Definition: The *nullity* of matrix A is $\dim \text{Null } A$

Rank-Nullity Theorem: For any n-column matrix A,

$$\text{nullity } A + \text{rank } A = n$$