[6] Dimension

The size of a basis

Key fact for this unit: all bases for a vector space have the same size.

We use this as the "basis" for answering many pending questions.

Morphing Lemma

Morphing Lemma: Suppose S is a set of vectors, and B is a linearly independent set of vectors in Span S. Then $|S| \geq |B|$.

Before we prove it—what good is this lemma?

Theorem: Any basis for $\mathcal{V}$ is a smallest generating set for $\mathcal{V}$.

Proof: Let S be a smallest generating set for $\mathcal{V}$. Let B be a basis for $\mathcal{V}$. Then B is a linearly independent set of vectors in Span S. By the Morphing Lemma, B is no bigger than S, so B is also a smallest generating set.

Theorem: All bases for a vector space $\mathcal{V}$ have the same size.

Proof: They are all smallest generating sets.

Proof of the Morphing Lemma

Morphing Lemma: Suppose S is a set of vectors, and B is a linearly independent set of vectors in Span S. Then $|S| \geq |B|$.

Proof outline: modify S step by step, introducing vectors of B one by one, without increasing the size.

How? Using the Exchange Lemma....

Review of Exchange Lemma

Exchange Lemma: Suppose S is a set of vectors and A is a subset of S. Suppose $\mathbf{z}$ is a vector in Span S such that $A \cup \{\mathbf{z}\}$ is linearly independent.
Then there is a vector $\mathbf{w} \in S - A$ such that

$$\text{Span } S = \text{Span } (S \cup \{\mathbf{z}\} - \{\mathbf{w}\})$$

Proof of the Morphing Lemma

Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Define $S_0 = S$.
Prove by induction on $k \leq n$ that there is a generating set S_k of Span S that contains $\mathbf{b}_1, \ldots, \mathbf{b}_k$ and has size $|S|$.

Base case: $k = 0$ is trivial.

To go from S_{k-1} to S_k: use the Exchange Lemma.

- $A_k = \{\mathbf{b}_1, \ldots, \mathbf{b}_{k-1}\}$ and $\mathbf{z} = \mathbf{b}_k$

Exchange Lemma $\Rightarrow$ there is a vector $\mathbf{w}$ in S_{k-1} such that

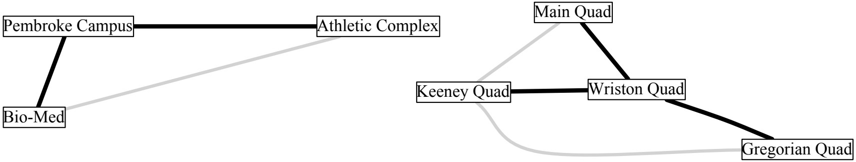
$$\text{Span } (S_{k-1} \cup \{\mathbf{b}_k\} - \{\mathbf{w}\}) = \text{Span } S_{k-1}$$

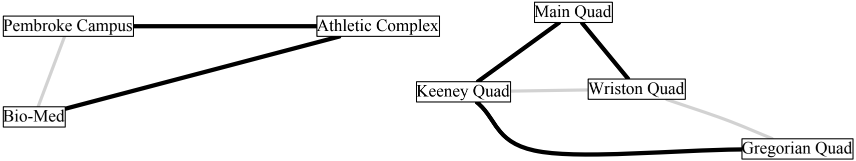
Set $S_k = S_{k-1} \cup \{\mathbf{b}_k\} - \{\mathbf{w}\}$.

QED

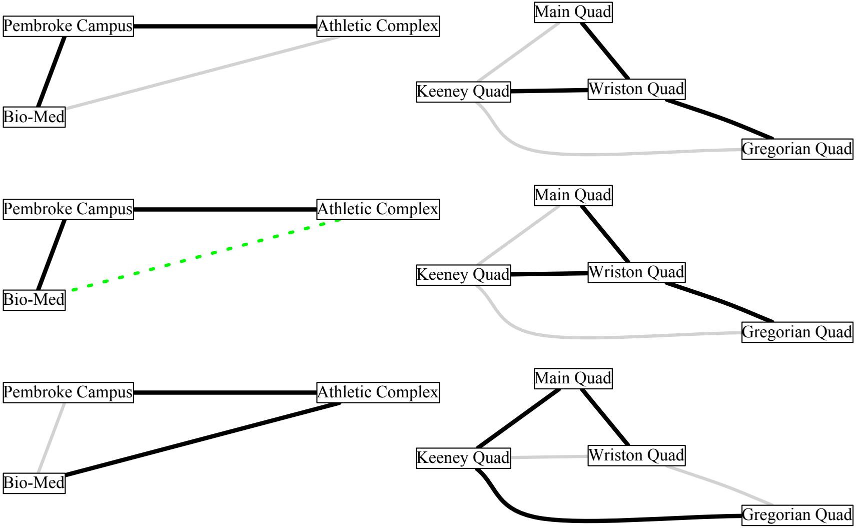
This induction proof is an algorithm.

Morphing from one spanning forest to another





Morphing from one spanning forest to another



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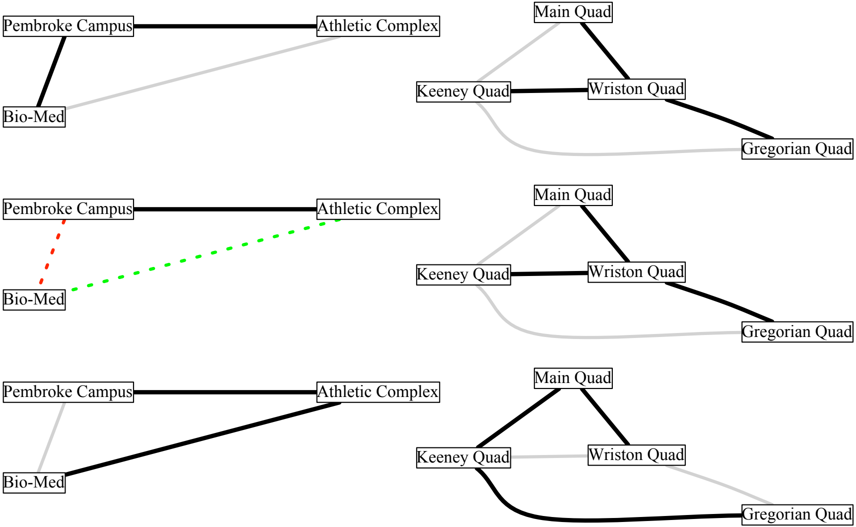

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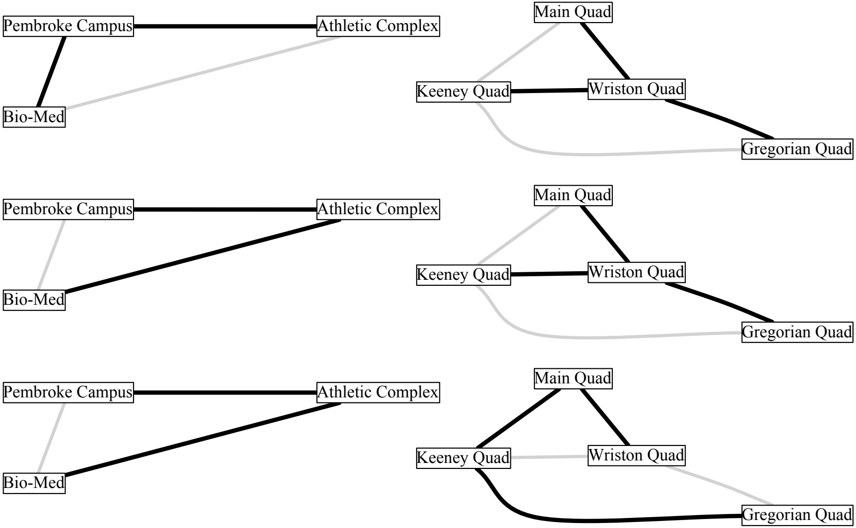

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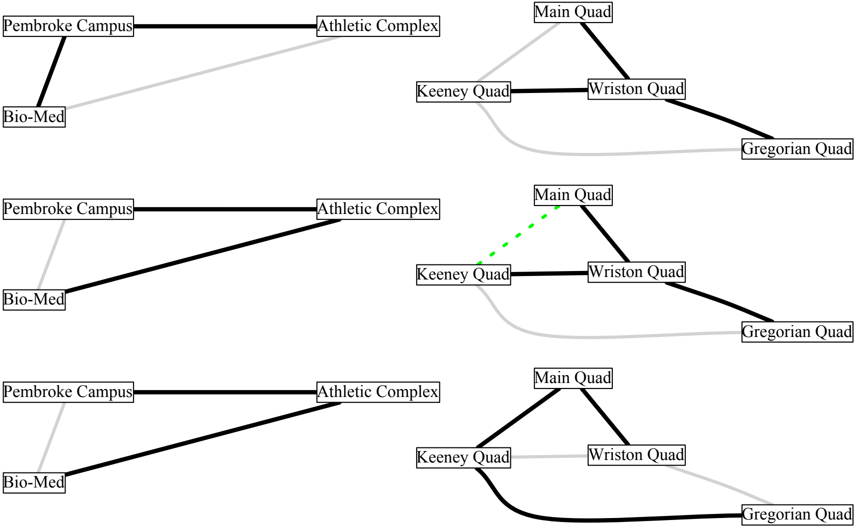

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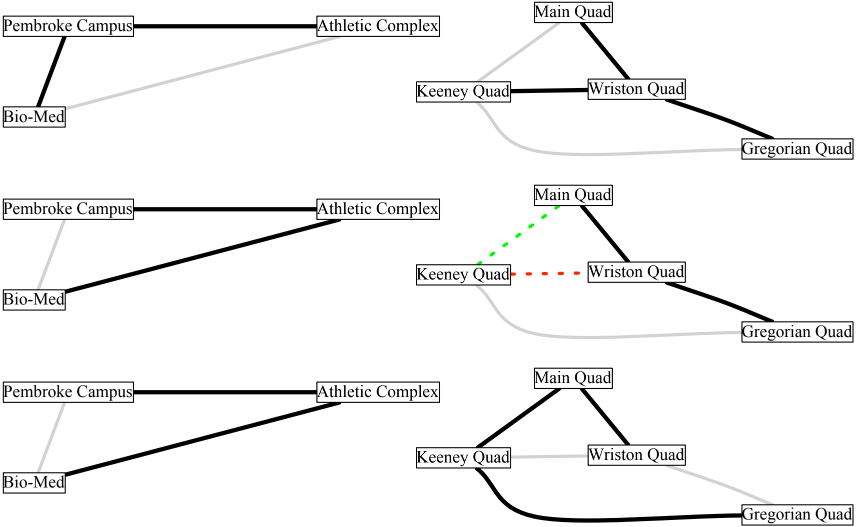

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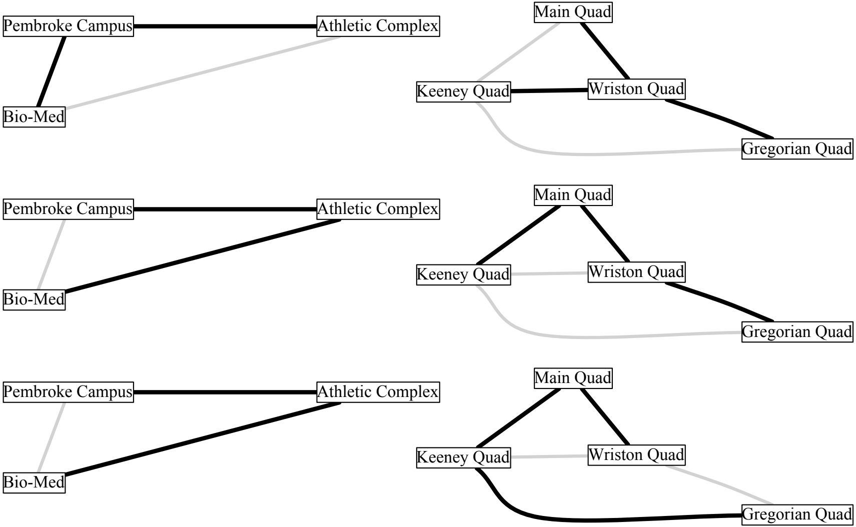

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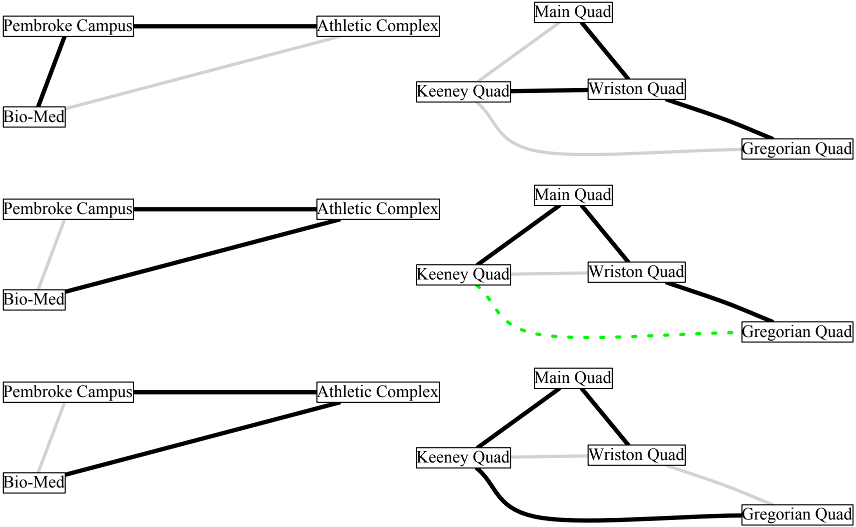
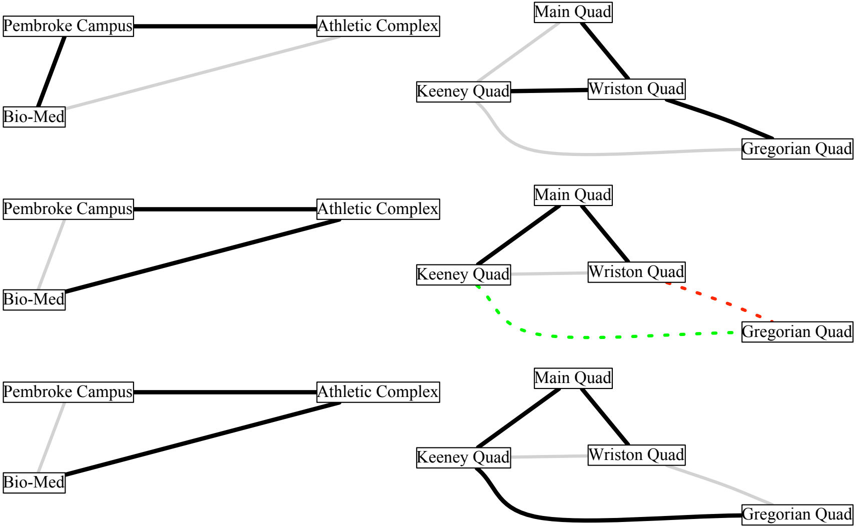
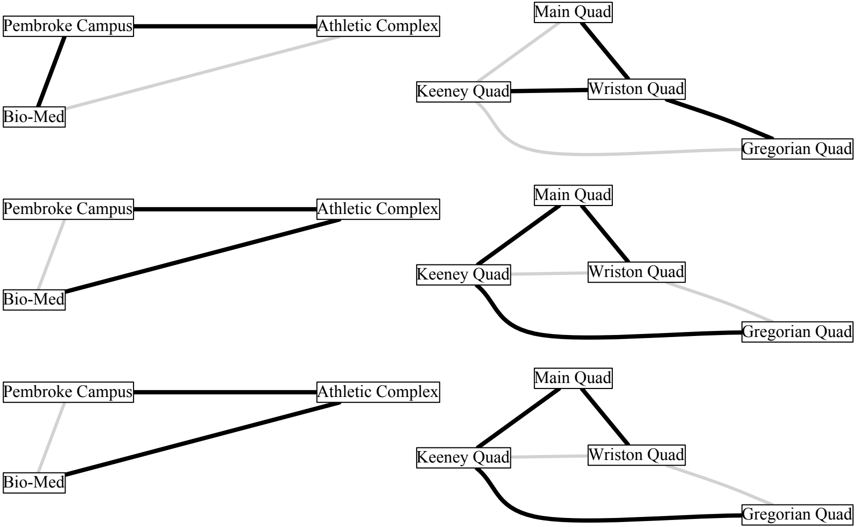


Morphing from one spanning forest to another



Morphing from one spanning forest to another



Dimension

Definition: Define *dimension* of a vector space $\mathcal{V}$ = size of a basis for $\mathcal{V}$. Written $\dim \mathcal{V}$.

Definition: Define *rank* of a set S of vectors = dimension of Span S. Written rank S.

Example: The vectors $[1, 0, 0], [0, 2, 0], [2, 4, 0]$ are linearly dependent.
Therefore their rank is less than three.
First two of these vectors form a basis for the span of all three, so the rank is two.

Example: The vector space Span $\{[0, 0, 0]\}$ is spanned by an empty set of vectors. Therefore the rank of $\{[0, 0, 0]\}$ is zero.

Row rank, column rank

Definition: For a matrix M, the *row rank* of M is the rank of its rows, and the *column rank* of M is the rank of its columns. Equivalently, row rank of M = dimension of Row M, and column rank of M = dimension of Col M.

Example: Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

whose rows are the vectors we saw before: $[1, 0, 0], [0, 2, 0], [2, 4, 0]$
The set of these vectors has rank two, so the row rank of M is two.
The columns of M are $[1, 0, 2]$, $[0, 2, 4]$, and $[0, 0, 0]$.
Since the third vector is the zero vector, it is not needed for spanning the column space.
Since each of the first two vectors has a nonzero where the other has a zero, these two are linearly independent, so the column rank is two.

Row rank, column rank

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Example: Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 2 & 0 & 7 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

Each of the rows has a nonzero where the others have zeroes, so the three rows are linearly independent. Thus the row rank of M is three.
The columns of M are $[1, 0, 0]$, $[0, 2, 0]$, $[0, 0, 3]$, and $[5, 7, 9]$.
The first three columns are linearly independent, and the fourth can be written as a linear combination of the first three, so the column rank is three.

Row rank, column rank

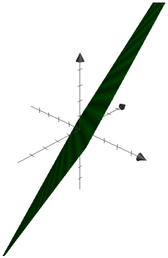
Definition: For a matrix M, the *row rank* of M is the rank of its rows, and the *column rank* of M is the rank of its columns. Equivalently, row rank of M = dimension of Row M, and column rank of M = dimension of Col M.

Does column rank always equal row rank? ☺

Geometry

We have asked:

Fundamental Question: How can we predict the dimensionality of the span of some vectors?



Now we can answer:

Compute the rank of the set of vectors.

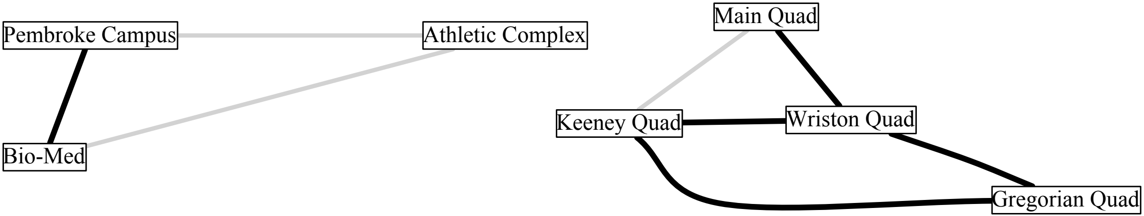
Examples:

• Span $\{[1, 2, -2]\}$ is a line but Span $\{[0, 0, 0]\}$ is a point.
First vector space has dimension one, second has dimension zero.

• Span $\{[1, 2], [3, 4]\}$ consists of all of $\mathbb{R}^2$ but Span $\{[1, 3], [2, 6]\}$ is a line
The first has dimension two and the second has dimension one.

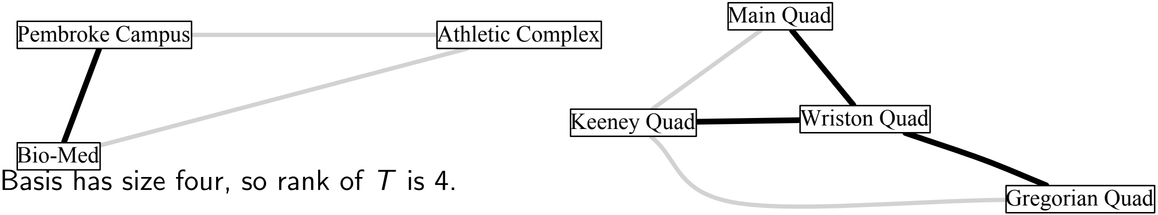
• Span $\{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}$ is $\mathbb{R}^3$ but Span $\{[1, 0, 0], [0, 1, 0], [1, 1, 0]\}$ is a plane.
The first has dimension three and the second has dimension two.

Dimension and rank in graphs



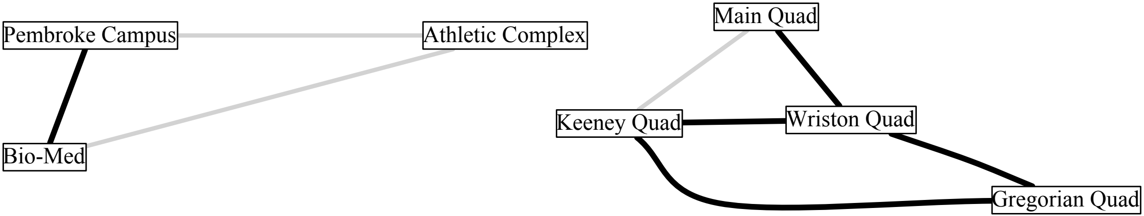
Let T = set of dark edges

Basis for Span T:



Basis has size four, so rank of T is 4.

Dimension and rank in graphs



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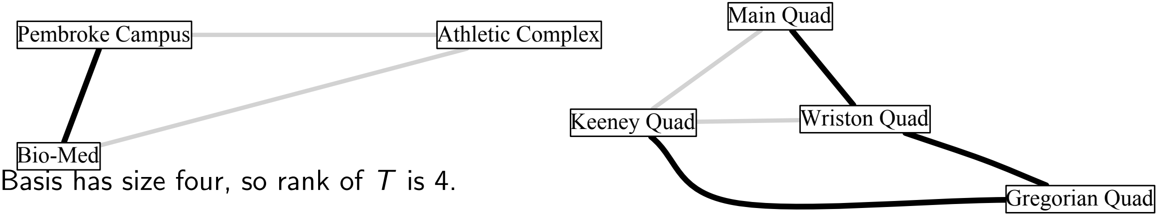

Basis has size four, so rank of T is 4.

Cardinality of a vector space over $GF(2)$

Cardinality of a vector space $\mathcal{V}$ over $GF(2)$ is $2^{\dim \mathcal{V}}$.

How to find dimension of solution set of a homogeneous linear system?

Write linear system as $A\mathbf{x} = \mathbf{0}$.

How to find dimension of the null space of A?

Answers will come later.

Subset-Basis Lemma

Lemma: Every finite set T of vectors contains a subset S that is a basis for Span T.

Proof: The Grow algorithm finds a basis for $\mathcal{V}$ if it terminates.

Initialize $S = \emptyset$. Repeat while possible: select a vector $\mathbf{v}$ in $\mathcal{V}$ that is not in Span S, and put it in S.

Revised version:

Initialize $S = \emptyset$ Repeat while possible: select a vector $\mathbf{v}$ in T that is not in Span S, and put it in S.

Differs from original:

- This algorithm stops when Span S contains every vector in T.
- The original Grow algorithm stops only once Span S contains every vector in $\mathcal{V}$.

However, that's okay: when Span S contains all the vectors in T, Span S also contains all linear combinations of vectors in T, so at this point Span $S = \mathcal{V}$.

Termination of Grow algorithm

def $\text{Grow}(\mathcal{V})$
  $B = \emptyset$
  repeat while possible:
    find a vector $\mathbf{v}$ in $\mathcal{V}$ that is not in Span B, and put it in B.

Grow-Algorithm-Termination Lemma: If $\mathcal{V}$ is a subspace of $\mathbb{F}^D$ where D is finite then $\text{Grow}(\mathcal{V})$ terminates.

Proof: By Grow-Algorithm Corollary, B is linearly independent throughout.

Apply the Morphing Lemma with $S = \{\text{standard generators for } \mathbb{F}^D\} \Rightarrow |B| \leq |S| = |D|$.

Since B grows in each iteration, there are at most $|D|$ iterations. QED

Every subspace of $\mathbb{F}^D$ contains a basis

Grow-Algorithm-Termination Lemma: If $\mathcal{V}$ is a subspace of $\mathbb{F}^D$ where D is finite then $\textsc{Grow}(\mathcal{V})$ terminates.

Theorem: For finite D, every subspace of $\mathbb{F}^D$ contains a basis.

Proof: Let $\mathcal{V}$ be a subspace of $\mathbb{F}^D$.

```
def GROW(V)
  B = ∅
  repeat while possible:
        find a vector v in V that is not in Span B, and put it in B.
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Grow-Algorithm-Termination Lemma ensures algorithm terminates.

Upon termination, every vector in $\mathcal{V}$ is in Span B, so B is a set of generators for $\mathcal{V}$. By Grow-Algorithm Corollary, B is linearly independent. Therefore B is a basis for $\mathcal{V}$. QED