Activity: Derive a matrix from input-output pairs

The 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies the following equations:

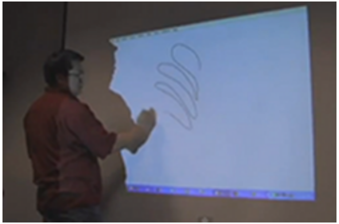
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 35 \\ 35 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Calculate the entries of the matrix.

Wiimote whiteboard

For location of infrared point, wiimote provides coordinate representation in terms of its camera basis).



Johnny Chung Lee, wiimote whiteboard

To use as a mouse, need to find corresponding location on screen (coordinate representation in tems of screen basis)

How to transform from one coordinate representation to the other?

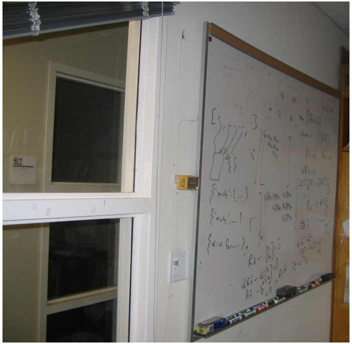
Can do this using a matrix H.

The challenge is to calculate the matrix H.

Can do this if you know the camera coordinate representation of four points whose screen coordinate representations are known.

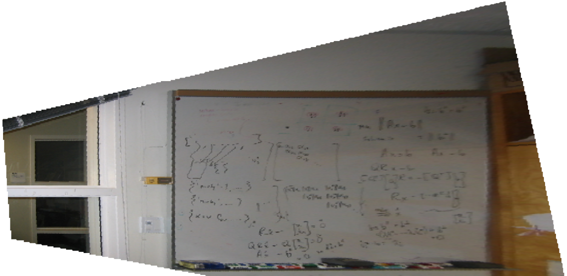
You'll do exactly the same computation but for a slightly different problem....

Removing perspective



Given an image of a whiteboard, taken from an angle...

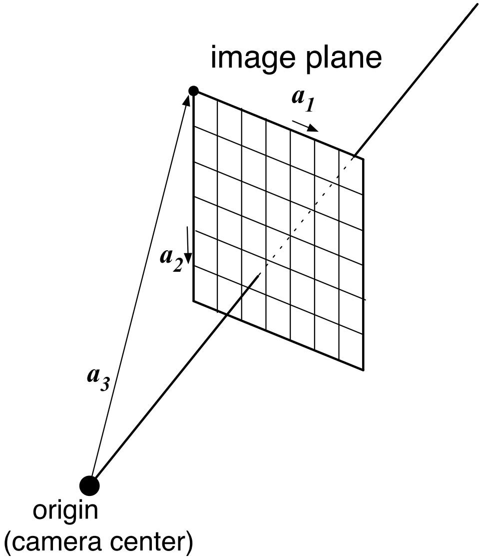
synthesize an image from straight ahead with no perspective



Camera coordinate system

We use same camera-oriented basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$:

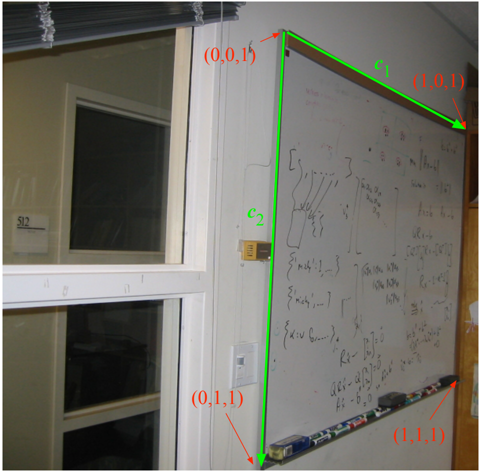
- The origin is the camera center.
- The first vector $\mathbf{a}_1$ goes horizontally from the top-left corner of a sensor element to the top-right corner.
- The second vector $\mathbf{a}_2$ goes vertically from the top-left corner of the sensor array to the bottom-left corner.
- The third vector $\mathbf{a}_3$ goes from the origin (the camera center) to the top-left corner of sensor element (0,0).



Converting from one basis to another

In addition, we define a *whiteboard basis* $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$

- The origin is the camera center.
- The first vector $\mathbf{c}_1$ goes horizontally from the top-left corner of whiteboard to top-right corner.
- The second vector $\mathbf{c}_2$ goes vertically from the top-left corner of whiteboard to the bottom-left corner.
- The third vector $\mathbf{c}_3$ goes from the origin (the camera center) to the top-left corner of whiteboard.



Converting between different basis representations

Start with a point $\mathbf{p}$ written in terms of in camera coordinates

$$\mathbf{p} = \left[\begin{array}{c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We write the same point $\mathbf{p}$ in the whiteboard coordinate system as

$$\mathbf{p} = \left[\begin{array}{c|c|c} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{array} \right] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Combining the two equations, we obtain

$$\left[\begin{array}{c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left[\begin{array}{c|c|c} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{array} \right] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Converting...

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Let A and C be the two matrices. As before, C has an inverse C^{-1}.
Multiplying equation on the left by C^{-1}, we obtain

$$\begin{bmatrix} & C^{-1} & \end{bmatrix} \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} & C^{-1} & \end{bmatrix} \begin{bmatrix} & C & \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Since C^{-1} and C cancel out, $\begin{bmatrix} & C^{-1} & \end{bmatrix} \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

We have shown that there is a matrix H (namely $H = C^{-1}A$) such that

$$\begin{bmatrix} & H & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

From pixel coordinates to whiteboard coordinates

		(x_1, x_2)	pixel coordinates
		$\downarrow$	
		$(x_1, x_2, 1)$	represent point in image plane in terms of camera basis
		$\downarrow$	
(y_1, y_2, y_3)	$=$	$H * (x_1, x_2, 1)$	change to representation in terms of whiteboard basis
$\downarrow$			
$(y_1/y_3, y_2/y_3, y_3/y_3)$			move to corresponding point in whiteboard plane
$\downarrow$			
$(y_1/y_3, y_2/y_3)$			get coordinates within whiteboard

Activity: Derive a matrix (up to a scale factor)

The 2×2 matrix A satisfies the following equations:

1. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ where $\begin{bmatrix} y_1/y_2 \\ y_2/y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
2. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} y_3 \\ y_4 \end{bmatrix}$ where $\begin{bmatrix} y_3/y_4 \\ y_4/y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
3. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} y_5 \\ y_6 \end{bmatrix}$ where $\begin{bmatrix} y_5/y_6 \\ y_6/y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Calculate the entries of the matrix *up to a scale factor*. That is, you are allowed to choose an arbitrary scale for the matrix. If your matrix is a scalar multiple of the true matrix, your answer is considered correct.

How to almost compute H

Write $H = \begin{bmatrix} h_{y_1,x_1} & h_{y_1,x_2} & h_{y_1,x_3} \\ h_{y_2,x_1} & h_{y_2,x_2} & h_{y_2,x_3} \\ h_{y_3,x_1} & h_{y_3,x_2} & h_{y_3,x_3} \end{bmatrix}$

The h_{ij}'s are the unknowns.

To derive equations, let **p** be some point on the whiteboard, and let **q** be the corresponding point on the image plane. Let $(x_1, x_2, 1)$ be the camera coordinates of **q**, and let (y_1, y_2, y_3) be the whiteboard coordinates of **q**. We have

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} h_{y_1,x_1} & h_{y_1,x_2} & h_{y_1,x_3} \\ h_{y_2,x_1} & h_{y_2,x_2} & h_{y_2,x_3} \\ h_{y_3,x_1} & h_{y_3,x_2} & h_{y_3,x_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

Multiplying out, we obtain

$$\begin{aligned} y_1 &= h_{y_1,x_1} x_1 + h_{y_1,x_2} x_2 + h_{y_1,x_3} \\ y_2 &= h_{y_2,x_1} x_1 + h_{y_2,x_2} x_2 + h_{y_2,x_3} \\ y_3 &= h_{y_3,x_1} x_1 + h_{y_3,x_2} x_2 + h_{y_3,x_3} \end{aligned}$$

Almost computing H

$$
\begin{aligned}
y_1 &= h_{y_1,x_1}x_1 + h_{y_1,x_2}x_2 + h_{y_1,x_3} \\
y_2 &= h_{y_2,x_1}x_1 + h_{y_2,x_2}x_2 + h_{y_2,x_3} \\
y_3 &= h_{y_3,x_1}x_1 + h_{y_3,x_2}x_2 + h_{y_3,x_3}
\end{aligned}
$$

Whiteboard coordinates of the original point $\mathbf{p}$ are $(y_1/y_3, y_2/y_3, 1)$. Define

$$
\begin{aligned}
w_1 &= y_1/y_3 \\
w_2 &= y_2/y_3
\end{aligned}
$$

so the whiteboard coordinates of $\mathbf{p}$ are $(w_1, w_2, 1)$.
Multiplying through by y_3, we obtain

$$
\begin{aligned}
w_1 y_3 &= y_1 \\
w_2 y_3 &= y_2
\end{aligned}
$$

Substituting our expressions for y_1, y_2, y_3, we obtain

$$
\begin{aligned}
w_1(h_{y_3,x_1}x_1 + h_{y_3,x_2}x_2 + h_{y_3,x_3}) = h_{y_1,x_1}x_1 + h_{y_1,x_2}x_2 + h_{y_1,x_3} \\
w_2(h_{y_3,x_1}x_1 + h_{y_3,x_2}x_2 + h_{y_3,x_3}) = h_{y_2,x_1}x_1 + h_{y_2,x_2}x_2 + h_{y_2,x_3}
\end{aligned}
$$

$$w_1(h_{y_3,x_1}x_1 + h_{y_3,x_2}x_2 + h_{y_3,x_3}) = h_{y_1,x_1}x_1 + h_{y_1,x_2}x_2 + h_{y_1,x_3}$$
$$w_2(h_{y_3,x_1}x_1 + h_{y_3,x_2}x_2 + h_{y_3,x_3}) = h_{y_2,x_1}x_1 + h_{y_2,x_2}x_2 + h_{y_2,x_3}$$

Multiplying through and moving everything to the same side, we obtain

$$(w_1x_1)h_{y_3,x_1} + (w_1x_2)h_{y_3,x_2} + w_1h_{y_3,x_3} - x_1h_{y_1,x_1} - x_2h_{y_1,x_2} - 1h_{y_1,x_3} = 0$$
$$(w_2x_1)h_{y_3,x_1} + (w_2x_2)h_{y_3,x_2} + w_2h_{y_3,x_3} - x_1h_{y_2,x_1} - x_2h_{y_2,x_2} - 1h_{y_2,x_3} = 0$$

Thus we get two linear equations in the unknowns. The coeffients are expressed in terms of x_1, x_2, w_1, w_2.

For four points, get eight equations. Need one more...

One more equation

We can't pin down H precisely.

This corresponds to the fact that we cannot recover the scale of the picture (a tiny building that is nearby looks just like a huge building that is far away).

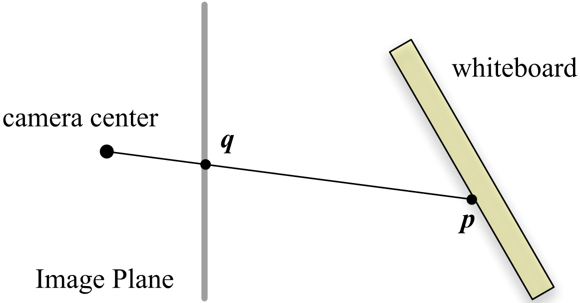
Fortunately, we don't need the true H.

As long as the H we compute is a scalar multiple of the true H, things will work out.

To arbitrarily select a scale, we add the equation $h_{y_1,x_1} = 1$.

Once you know H

1. For each point **q** in the representation of the image, we have the camera coordinates $(x_1, x_2, 1)$ of **q**. We multiply by H to obtain the whiteboard coordinates (y_1, y_2, y_3) of the same point **q**.
2. Recall the situation as viewed from above:



The whiteboard coordinates of the corresponding point **p** on the whiteboard are $(y_1/y_3, y_2/y_3, 1)$. Use this formula to compute these coordinates.
3. Display the updated points with the same color matrix

Quiz

Draw diagrams showing

- one way in which a subset of a cartesian product $A \times B$ can fail to be a function from A to B, and
- a second way;
- one way in which a function from A to B can fail to be invertible,
- a second way;

Simplified Exchange Lemma

We need a tool to iteratively transform one set of generators into another.

- You have a set S of vectors.
- You have a vector z you want to inject into S.
- You want to maintain same size so must eject a vector from S.
- You want the span to not change.

Exchange Lemma tells you how to choose vector to eject.

Simplified Exchange Lemma:

- Suppose S is a set of vectors.
- Suppose $\mathbf{z}$ is a nonzero vector in Span S.
- Then there is a vector $\mathbf{w}$ in S such that

$$\text{Span } (S \cup \{\mathbf{z}\} - \{\mathbf{w}\}) = \text{Span } S$$

Simplified Exchange Lemma proof

Simplified Exchange Lemma: Suppose S is a set of vectors, and $\mathbf{z}$ is a nonzero vector in Span S. Then there is a vector $\mathbf{w}$ in S such that Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\}) =$ Span S.

Proof: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Since $\mathbf{z}$ is in Span S, can write

$$\mathbf{z} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

By Superfluous-Vector Lemma, Span $(S \cup \{z\}) =$ Span S.
Since $\mathbf{z}$ is nonzero, at least one of the coefficients is nonzero, say α_i.
Rewrite as

$$\mathbf{z} - \alpha_1 \mathbf{v}_1 - \cdots - \alpha_{i-1} \mathbf{v}_{i-1} - \alpha_{i+1} \mathbf{v}_{i+1} - \cdots - \alpha_n \mathbf{v}_n = \alpha_i \mathbf{v}_i$$

Divide through by α_i:

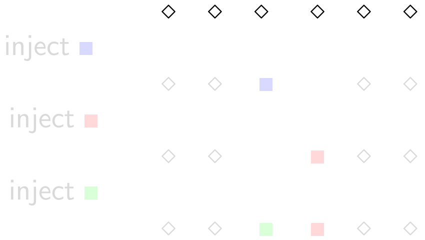
$$(1/\alpha_i)\mathbf{z} - (\alpha_1/\alpha_i) \mathbf{v}_1 - \cdots - (\alpha_{i-1}/\alpha_i) \mathbf{v}_{i-1} - (\alpha_{i+1}/\alpha_i) \mathbf{v}_{i+1} - \cdots - (\alpha_n/\alpha_i) \mathbf{v}_n = \mathbf{v}_i$$

By Superfluous-Vector Lemma, Span $(S \cup \{\mathbf{z}\}) =$ Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\})$. QED

Exchange Lemma

Simplified Exchange Lemma: Suppose S is a set of vectors, and $\mathbf{z}$ is a nonzero vector in Span S. Then there is a vector $\mathbf{w}$ in S such that
Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\}) = \text{Span } S$.

Simplified Exchange Lemma helps in transforming one generating set into another...



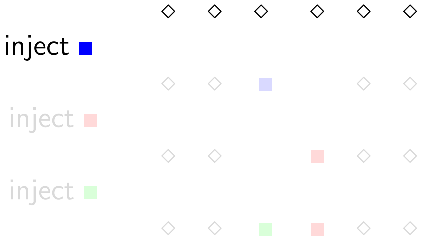
Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square

Need a way to protect some elements from being taken out.

Exchange Lemma

Simplified Exchange Lemma: Suppose S is a set of vectors, and $\mathbf{z}$ is a nonzero vector in Span S. Then there is a vector $\mathbf{w}$ in S such that Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\}) = \text{Span } S$.

Simplified Exchange Lemma helps in transforming one generating set into another...



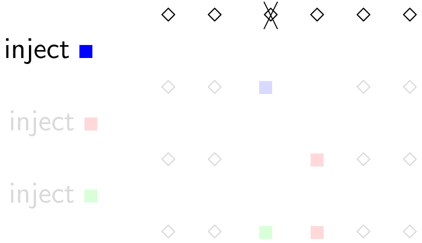
Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square

Need a way to protect some elements from being taken out.

Exchange Lemma

Simplified Exchange Lemma: Suppose S is a set of vectors, and $\mathbf{z}$ is a nonzero vector in Span S. Then there is a vector $\mathbf{w}$ in S such that
Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\}) = \text{Span } S$.

Simplified Exchange Lemma helps in transforming one generating set into another...



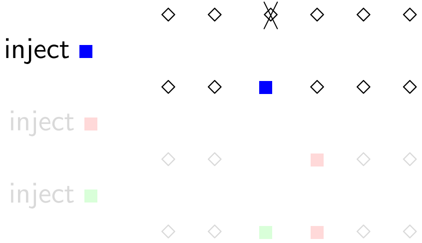
Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square

Need a way to protect some elements from being taken out.

Exchange Lemma

Simplified Exchange Lemma: Suppose S is a set of vectors, and $\mathbf{z}$ is a nonzero vector in Span S. Then there is a vector $\mathbf{w}$ in S such that
Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\}) = \text{Span } S$.

Simplified Exchange Lemma helps in transforming one generating set into another...



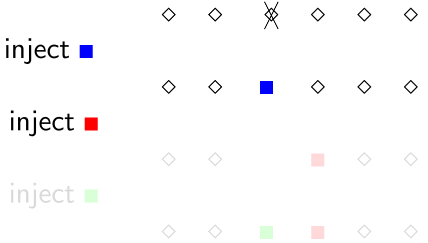
Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square

Need a way to protect some elements from being taken out.

Exchange Lemma

Simplified Exchange Lemma: Suppose S is a set of vectors, and $\mathbf{z}$ is a nonzero vector in Span S. Then there is a vector $\mathbf{w}$ in S such that Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\}) = $ Span S.

Simplified Exchange Lemma helps in transforming one generating set into another...



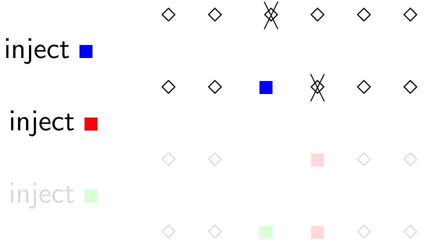
Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square

Need a way to protect some elements from being taken out.

Exchange Lemma

Simplified Exchange Lemma: Suppose S is a set of vectors, and $\mathbf{z}$ is a nonzero vector in Span S. Then there is a vector $\mathbf{w}$ in S such that Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\}) = $ Span S.

Simplified Exchange Lemma helps in transforming one generating set into another...



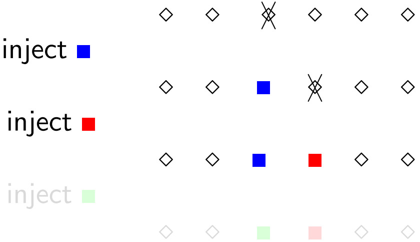
Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square

Need a way to protect some elements from being taken out.

Exchange Lemma

Simplified Exchange Lemma: Suppose S is a set of vectors, and $\mathbf{z}$ is a nonzero vector in Span S. Then there is a vector $\mathbf{w}$ in S such that Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\}) = $ Span S.

Simplified Exchange Lemma helps in transforming one generating set into another...



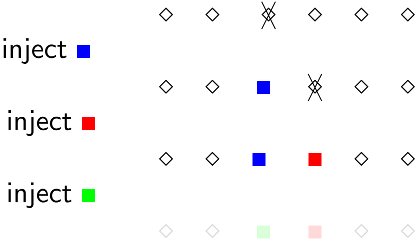
Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square

Need a way to protect some elements from being taken out.

Exchange Lemma

Simplified Exchange Lemma: Suppose S is a set of vectors, and $\mathbf{z}$ is a nonzero vector in Span S. Then there is a vector $\mathbf{w}$ in S such that
Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\}) =$ Span S.

Simplified Exchange Lemma helps in transforming one generating set into another...



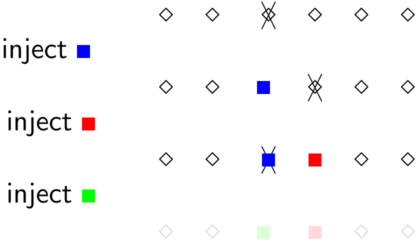
Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square

Need a way to protect some elements from being taken out.

Exchange Lemma

Simplified Exchange Lemma: Suppose S is a set of vectors, and $\mathbf{z}$ is a nonzero vector in Span S. Then there is a vector $\mathbf{w}$ in S such that Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\}) = \text{Span } S$.

Simplified Exchange Lemma helps in transforming one generating set into another...



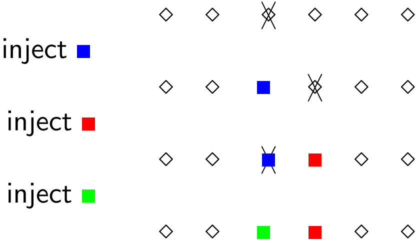
Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square

Need a way to protect some elements from being taken out.

Exchange Lemma

Simplified Exchange Lemma: Suppose S is a set of vectors, and $\mathbf{z}$ is a nonzero vector in Span S. Then there is a vector $\mathbf{w}$ in S such that Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\})$ = Span S.

Simplified Exchange Lemma helps in transforming one generating set into another...



Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square

Need a way to protect some elements from being taken out.

Exchange Lemma

Simplified Exchange Lemma: Suppose S is a set of vectors, and $\mathbf{z}$ is a nonzero vector in Span S. Then there is a vector $\mathbf{w}$ in S such that Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\}) = \text{Span } S$.

Need to enhance this lemma. Set of *protected* elements is A:

Exchange Lemma:

- Suppose S is a set of vectors and A is a subset of S.
- Suppose $\mathbf{z}$ is a vector in Span S such that $A \cup \{\mathbf{z}\}$ is linearly independent.
- Then there is a vector $\mathbf{w} \in S - A$ such that Span $S = \text{Span } (S \cup \{\mathbf{z}\} - \{\mathbf{w}\})$

Now, not enough that $\mathbf{z}$ be nonzero—need A to be linearly independent.

Exchange Lemma proof

Exchange Lemma: Suppose S is a set of vectors and A is a subset of S. Suppose $\mathbf{z}$ is a vector in Span S such that $A \cup \{\mathbf{z}\}$ is linearly independent.
Then there is a vector $\mathbf{w} \in S - A$ such that Span $S =$ Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\})$

Proof: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ and $A = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.
Since $\mathbf{z}$ is in Span S, can write

$$\mathbf{z} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k + \beta_1 \mathbf{w}_1 + \cdots + \beta_\ell \mathbf{w}_\ell$$

By Superfluous-Vector Lemma, Span $(S \cup \{z\}) =$ Span S.
If coefficients $\beta_1, \ldots, \beta_\ell$ were all zero then we would have $\mathbf{z} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k$, contradicting the linear independence of $A \cup \{\mathbf{z}\}$.
Thus one of the coefficients $\beta_1, \ldots, \beta_\ell$ must be nonzero... say β_1. Rewrite as

$$\mathbf{z} - \alpha_1 \mathbf{v}_1 - \cdots - \alpha_k \mathbf{v}_k - \beta_2 \mathbf{w}_2 - \cdots - \beta_\ell \mathbf{w}_\ell = \beta_1 \mathbf{w}_1$$

Divide through by β_1:

$$(1/\beta_1)\mathbf{z} - (\alpha_1/\beta_1) \mathbf{v}_1 - \cdots - (\alpha_k/\beta_1) \mathbf{v}_k - (\beta_2/\beta_1) \mathbf{w}_2 - \cdots - (\beta_\ell/\beta_1) \mathbf{w}_\ell = \mathbf{w}_1$$

By Superfluous-Vector Lemma, Span $(S \cup \{\mathbf{z}\}) =$ Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}_1\})$. QED