Unique representation

Recall idea of *coordinate system* for a vector space $\mathcal{V}$:

- Generators $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of $\mathcal{V}$
- Every vector $\mathbf{v}$ in $\mathcal{V}$ can be written as a linear combination
$$\mathbf{v} = \alpha_1 \, \mathbf{a}_1 + \cdots + \alpha_n \, \mathbf{a}_n$$
- We represent vector $\mathbf{v}$ by its *coordinate representation* $[\alpha_1, \ldots, \alpha_n]$

Question: How can we ensure that each point has only one coordinate representation?

Answer: The generators $\mathbf{a}_1, \ldots, \mathbf{a}_n$ should form a basis.

> **Unique-Representation Lemma** Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be a basis for $\mathcal{V}$. For any vector $\mathbf{v} \in \mathcal{V}$, there is exactly one representation of $\mathbf{v}$ in terms of the basis vectors.

Uniqueness of representation in terms of a basis

Unique-Representation Lemma: Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be a basis for $\mathcal{V}$. For any vector $\mathbf{v} \in \mathcal{V}$, there is exactly one representation of $\mathbf{v}$ in terms of the basis vectors.

Proof: Let $\mathbf{v}$ be any vector in $\mathcal{V}$.

The vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ span $\mathcal{V}$, so there is at least one representation of $\mathbf{v}$ in terms of the basis vectors.

Suppose there are two such representations:

$$\mathbf{v} = \alpha_1 \, \mathbf{a}_1 + \cdots + \alpha_n \, \mathbf{a}_n = \beta_1 \, \mathbf{a}_1 + \cdots + \beta_n \, \mathbf{a}_n$$

We get the zero vector by subtracting one from the other:

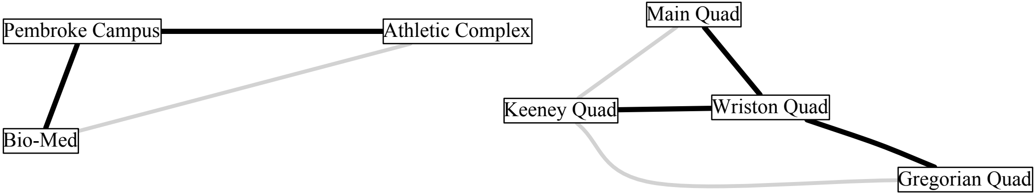
$$\begin{aligned} \mathbf{0} &= \alpha_1 \, \mathbf{a}_1 + \cdots + \alpha_n \, \mathbf{a}_n - (\beta_1 \, \mathbf{a}_1 + \cdots + \beta_n \, \mathbf{a}_n) \\ &= (\alpha_1 - \beta_1) \, \mathbf{a}_1 + \cdots + (\alpha_n - \beta_n) \, \mathbf{a}_n \end{aligned}$$

Since the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent, the coefficients $\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n$ must all be zero, so the two representations are really the same.

QED

Uniqueness of representation in terms of a basis: The case of graphs

Unique-Representation Lemma Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be a basis for $\mathcal{V}$. For any vector $\mathbf{v} \in \mathcal{V}$, there is exactly one representation of $\mathbf{v}$ in terms of the basis vectors.



A basis for a graph is a spanning forest.

Unique Representation shows that, for each edge xy in the graph,

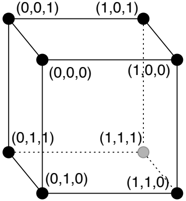
- there is an x-to-y path in the spanning forest, and
- there is only one such path.

Change of basis

Examples:

Lossy compression of images (described earlier) or audio

Perspective rendering



Removing perspective from an image

Change of basis

Suppose $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is a basis for $\mathcal{V}$. How do we go

- from a vector $\mathbf{b}$ in $\mathcal{V}$
- to the coordinate representation $\mathbf{u}$ of $\mathbf{b}$ in terms of $\mathbf{a}_1, \ldots, \mathbf{a}_n$?

By linear-comb. definition of matrix-vector mult.,

$$\left[\begin{array}{c|c|c} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{array}\right] \left[\begin{array}{c} \mathbf{u} \end{array}\right] = \left[\begin{array}{c} \mathbf{b} \end{array}\right]$$

By Unique-Representation Lemma,
$\mathbf{u}$ is the *only* solution to the equation

$$\left[\begin{array}{c|c|c} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{array}\right] \left[\begin{array}{c} \mathbf{x} \end{array}\right] = \left[\begin{array}{c} \mathbf{b} \end{array}\right]$$

so we can obtain $\mathbf{u}$ by solving a matrix-vector equation.

Important special case: $\mathcal{V} = \mathbb{F}^m$.

Function $f : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ defined by $f(\mathbf{x}) = \left[\begin{array}{c|c|c} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{array}\right] \left[\begin{array}{c} \mathbf{x} \end{array}\right]$ is

- *onto* (because $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are generators for $\mathbb{F}^m$)
- *one-to-one* (by Unique-Representation Lemma)

so f is an invertible function
so the matrix is invertible.

Change of basis

Now suppose $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is one basis for $\mathcal{V}$ and $\mathbf{c}_1, \cdots, \mathbf{c}_k$ is another.

Define $f(\mathbf{x}) = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix}$ and define $g(\mathbf{y}) = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k \end{bmatrix} \begin{bmatrix} \mathbf{y} \end{bmatrix}$.

Then both f and g are invertible functions.

The function $f^{-1} \circ g$ maps

- from coordinate representation of a vector in terms of $\mathbf{c}_1, \ldots, \mathbf{c}_k$
- to coordinate representation of a vector in terms of $\mathbf{a}_1, \ldots, \mathbf{a}_n$

In particular, if $\mathcal{V} = \mathbb{F}^m$ for some m then

f invertible implies that $\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ is an invertible matrix.

g invertible implies that $\begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k \end{bmatrix}$ is an invertible matrix.

Thus the function $f^{-1} \circ g$ has the property

$$(f^{-1} \circ g)(\mathbf{x}) = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix}$$

Change of basis

Proposition: If $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{c}_1, \ldots, \mathbf{c}_k$ are bases for $\mathbb{F}^m$ then multiplication by the matrix

$$B = \left[\begin{array}{c|c|c} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{array} \right]^{-1} \left[\begin{array}{c|c|c} \mathbf{c}_1 & \cdots & \mathbf{c}_k \end{array} \right]$$

maps

- from the coordinate representation of a vector with respect to $\mathbf{c}_1, \ldots, \mathbf{c}_k$
- to the coordinate representation of that vector with respect to $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Conclusion: Given two bases of $\mathbb{F}^m$, there is a matrix B such that multiplication by B converts from one coordinate representation to the other.

Remark: Converting between vector itself and its coordinate representation is a special case:

- Think of the vector itself as coordinate representation with respect to standard basis.

Change of basis: simple example

Example: To map

from coordinate representation with respect to $[1, 2, 3], [2, 1, 0], [0, 1, 4]$

to coordinate representation with respect to $[2, 0, 1], [0, 1, -1], [1, 2, 0]$

multiply by the matrix

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & 4 \end{bmatrix}$$

which is

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{4}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & 4 \end{bmatrix}$$

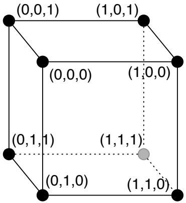
which is

$$\begin{bmatrix} -1 & 1 & -\frac{5}{3} \\ -4 & 1 & -\frac{17}{3} \\ 3 & 0 & \frac{10}{3} \end{bmatrix}$$

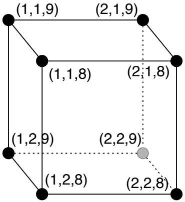
Perspective rendering

As application of change of basis, we show how to synthesize a camera view from a set of points in three dimensions, taking into account perspective.
The math will be useful in next lab, where we will go in the opposite direction, removing perspective from a real image.

We start with the points making up a wire cube:



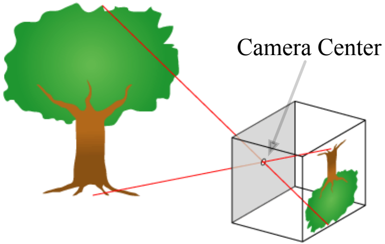
For reasons that will become apparent, we translate the cube, adding $(1, 1, 8)$ to each point.



How does a camera (or an eye) see these points?

Simplified camera model

Simplified model of a camera:

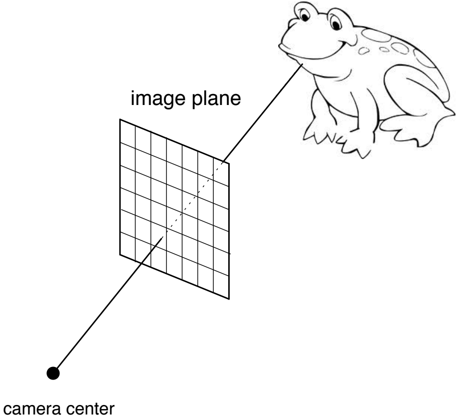


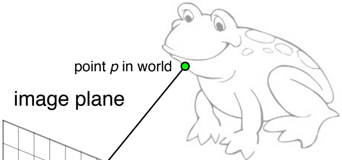
- There is a point called the *camera center*.
- There is an image sensor array in the back of the camera.
- Photons bounce off objects in the scene and travel through the camera center to the image sensor array.
- A photon from the scene only reaches the image sensor array if it travels in a straight line through the camera center.
- The image ends up being inverted.

Even more simplified camera model

Even simpler model to avoid the inversion:

- The image sensor array is between the camera center and the scene.
- The image sensor array is located in a plane, called the *image plane*.
- A photon from the scene is detected by the sensor array only if it is traveling in a straight line towards the camera center.
- The sensor element that detects the photon is the one intersected by this line.
- Need a function that maps from point **p** in world to corresponding point **q** in image plane

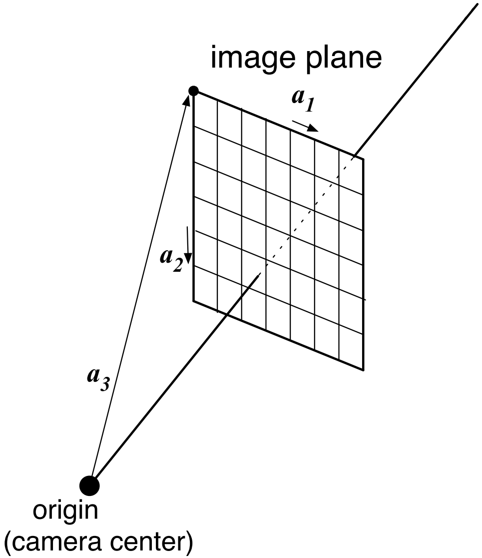




Camera coordinate system

Camera-oriented basis helps in mapping from world points to image-plane points:

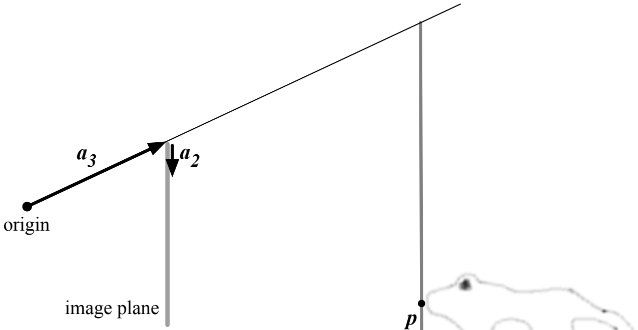
- The origin is defined to be the camera center. (That's why we translated the wire-frame cube.)
- The first vector $\mathbf{a}_1$ goes horizontally from the top-left corner of a sensor element to the top-right corner.
- The second vector $\mathbf{a}_2$ goes vertically from the top-left corner of a sensor element to the bottom-left corner.
- The third vector $\mathbf{a}_3$ goes from the origin (the camera center) to the top-left corner of sensor element (0,0).



From world point to camera-plane point

Side view (we see only the edge of the image plane)

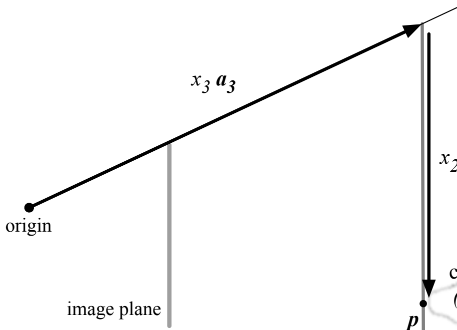
- Have a point **p** in the world
- Express it in terms of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$
- Consider corresponding point **q** in image plane.
- Similar triangles $\Rightarrow$ coordinates of **q**



Summary: Given coordinate representation (x_1, x_2, x_3) in terms of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$,

coordinate representation of corresponding point in image plane is $(x_1/x_3, x_2/x_3, x_3/x_3)$.

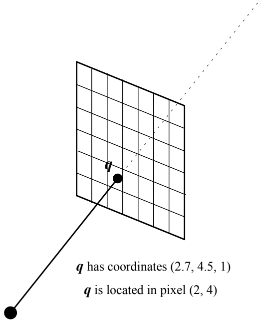
I call this *scaling down*.



Converting to pixel coordinates

Converting from a point (x_1, x_2, x_3) in the image plane to pixel coordinates

- Drop third entry x_3 (it is always equal to 1)



From world coordinates to camera coordinates to pixel coordinates

Write basis vectors of camera coordinate system using world coordinates

For each point $\mathbf{p}$ in the wire-frame cube,

- find representation in $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$
- scale down to get corresponding point in image plane
- convert to pixel coordinates by dropping third entry x_3