Activity

Suppose you have available a procedure `is_independent(L)`, which takes a list `L` of `Vec`s and returns True or False depending on whether the vectors are independent or not.

Write a procedure

```
def has_many_solutions(a_list, b_list, u)...
```

with the following spec:

- **input:** list $[\mathbf{a}_1, \ldots, \mathbf{a}_n]$ of Vecs, list $[\beta_1, \ldots, \beta_n]$ of scalars, Vec $\mathbf{u}$ that is a solution to the linear system
$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= \beta_1 \\ &\vdots \\ \mathbf{a}_n \cdot \mathbf{x} &= \beta_n \end{aligned}$$
- **output:** `True` if there are solutions other than $\mathbf{u}$ to the linear system.

Activity

So far we've done *paths* $=$ *spanning* and *cycles* $=$ *linearly dependent* over GF(2). How would you achieve the same over $\mathbb{R}$?

Properties of linear (in)dependence

Linear-Dependence Lemma Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors. A vector $\mathbf{v}_i$ is in the span of the other vectors if and only if the zero vector can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in which the coefficient of $\mathbf{v}_i$ is nonzero.

Contrapositive:

$\mathbf{v}_i$ is *not* in the space of the other vectors
if and only if
for any linear combination equaling the zero vector

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_i \mathbf{v}_i + \cdots + \alpha_n \mathbf{v}_n$$

it must be that the coefficient α_i *is* zero.

Analyzing the Grow algorithm

def $\text{Grow}(\mathcal{V})$
  $S = \emptyset$
  repeat while possible:
    find a vector $\mathbf{v}$ in $\mathcal{V}$ that is not in Span S, and put it in S.

Grow-Algorithm Corollary: The vectors obtained by the Grow algorithm are linearly independent.

In graphs, this means that the solution obtained by the Grow algorithm has no cycles (is a forest).

Analyzing the Grow algorithm

Grow-Algorithm Corollary: The vectors obtained by the Grow algorithm are linearly independent.

Proof: For $n = 1, 2, \ldots$, let $\mathbf{v}_n$ be the vector added to S in the n^{th} iteration of the Grow algorithm. We show by induction that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.

For $n = 0$, there are no vectors, so the claim is trivially true.
Assume the claim is true for $n = k - 1$. We prove it for $n = k$.
The vector $\mathbf{v}_k$ added to S in the k^{th} iteration is not in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$.
Therefore, by the Linear-Dependence Lemma, for any coefficients $\alpha_1, \ldots, \alpha_k$ such that

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_{k-1} \mathbf{v}_{k-1} + \alpha_k \mathbf{v}_k$$

it must be that α_k equals zero. We may therefore write

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_{k-1} \mathbf{v}_{k-1}$$

By claim for $n = k - 1$, $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$ are linearly independent, so $\alpha_1 = \cdots = \alpha_{k-1} = 0$

The linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is *trivial*. We have proved that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent. This proves the claim for $n = k$. QED

Analyzing the Shrink algorithm

def $\text{Shrink}(\mathcal{V})$
    $S =$ some finite set of vectors that spans $\mathcal{V}$
    repeat while possible:
        find a vector **v** in S such that Span $(S - \{v\}) = \mathcal{V}$, and remove **v** from S.

Shrink-Algorithm Corollary: The vectors obtained by the Shrink algorithm are linearly independent.

In graphs, this means that the Shrink algorithm outputs a solution that is a forest.

Recall:

Superfluous-Vector Lemma For any set S and any vector $\mathbf{v} \in S$, if **v** can be written as a linear combination of the other vectors in S then
Span $(S - \{\mathbf{v}\}) =$ Span S

Analyzing the Shrink algorithm

Shrink-Algorithm Corollary: The vectors obtained by the Shrink algorithm are linearly independent.

Proof: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be the set of vectors obtained by the Shrink algorithm. Assume for a contradiction that the vectors are linearly dependent.

Then $\mathbf{0}$ can be written as a nontrivial linear combination

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

where at least one of the coefficients is nonzero.

Let α_i be one of the nonzero coefficients.

By the Linear-Dependence Lemma, $\mathbf{v}_i$ can be written as a linear combination of the other vectors.

Hence by the Superfluous-Vector Lemma, Span $(S - \{\mathbf{v}_i\}) =$ Span S,

so the Shrink algorithm should have removed $\mathbf{v}_i$. QED

Basis

If they successfully finish, the Grow algorithm and the Shrink algorithm each find a set of vectors spanning the vector space $\mathcal{V}$. In each case, the set of vectors found is linearly independent.

Definition: Let $\mathcal{V}$ be a vector space. A *basis* for $\mathcal{V}$ is a linearly independent set of generators for $\mathcal{V}$.

Thus a set S of vectors of $\mathcal{V}$ is a *basis* for $\mathcal{V}$ if S satisfies two properties:

Property B1 (*Spanning*) Span $S = \mathcal{V}$, and

Property B2 (*Independent*) S is linearly independent.

Most important definition in linear algebra.

Basis: Examples

A set S of vectors of $\mathcal{V}$ is a *basis* for $\mathcal{V}$ if S satisfies two properties:

Property B1 (*Spanning*) Span $S = \mathcal{V}$, and

Property B2 (*Independent*) S is linearly independent.

Example: Let $\mathcal{V} = \text{Span } \{[1, 0, 2, 0], [0, -1, 0, -2], [2, 2, 4, 4]\}$.

Is $\{[1, 0, 2, 0], [0, -1, 0, -2], [2, 2, 4, 4]\}$ a basis for $\mathcal{V}$?

The set *is* spanning but is *not* independent

$$1\,[1, 0, 2, 0] - 1\,[0, -1, 0, -2] - \frac{1}{2}\,[2, 2, 4, 4] = \mathbf{0}$$

so not a basis

However, $\{[1, 0, 2, 0], [0, -1, 0, -2]\}$ *is* a basis:

- Obvious that these vectors are independent because each has a nonzero entry where the other has a zero.
- To show

$$\text{Span } \{[1, 0, 2, 0], [0, -1, 0, -2]\} = \text{Span } \{[1, 0, 2, 0], [0, -1, 0, -2], [2, 2, 4, 4]\},$$

  can use Superfluous-Vector Lemma:

$$[2, 2, 4, 4] = 2\,[1, 0, 2, 0] - 2\,[0, -1, 0, -2]$$

Basis: Examples

Example: A simple basis for $\mathbb{R}^3$: the standard generators $\mathbf{e}_1 = [1,0,0], \mathbf{e}_2 = [0,1,0], \mathbf{e}_3 = [0,0,1]$.

- *Spanning:* For any vector $[x,y,z] \in \mathbb{R}^3$,

$$[x,y,z] = x\,[1,0,0] + y\,[0,1,0] + z\,[0,0,1]$$

- *Independent:* Suppose

$$\mathbf{0} = \alpha_1\,[1,0,0] + \alpha_2\,[0,1,0] + \alpha_3\,[0,0,1] = [\alpha_1,\alpha_2,\alpha_3]$$

  Then $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Instead of "standard generators", we call them *standard basis vectors*.
We refer to $\{[1,0,0],[0,1,0],[0,0,1]\}$ as *standard basis* for $\mathbb{R}^3$.

In general the standard generators are usually called *standard basis vectors*.

Basis: Examples

Example: Another basis for $\mathbb{R}^3$: $[1,1,1],[1,1,0],[0,1,1]$

- *Spanning:* Can write standard generators in terms of these vectors:

$$\begin{aligned}[1,0,0] &= [1,1,1]-[0,1,1]\\ [0,1,0] &= [1,1,0]+[0,1,1]-[1,1,1]\\ [0,0,1] &= [1,1,1]-[1,1,0]\end{aligned}$$

  Since $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ can be written in terms of these new vectors, every vector in Span $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is in span of new vectors. Thus $\mathbb{R}^3$ equals span of new vectors.

- *Linearly independent:* Write zero vector as linear combination:

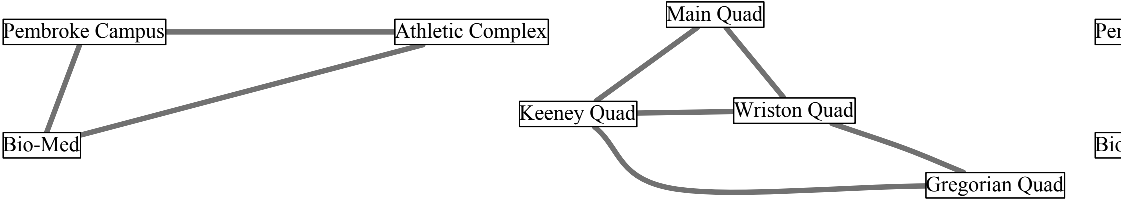
$$\mathbf{0} = x\,[1,1,1] + y\,[1,1,0] + z\,[0,1,1] = [x+y, x+y+z, x+z]$$

Looking at each entry, we get

$$\begin{aligned}0 &= x+y\\ 0 &= x+y+z\\ 0 &= x+z\end{aligned}$$

Plug $x+y=0$ into second equation to get $0=z$.
Plug $z=0$ into third equation to get $x=0$.
Plug $x=0$ into first equation to get $y=0$.
Thus the linear combination is trivial.

Basis: Examples in graphs



One kind of basis in a graph G: a set S of edges forming a spanning forest.

- *Spanning:* for each edge xy in G, there is an x-to-y path consisting of edges of S.
- *Independent:* no cycle consisting of edges of S

Towards showing that every vector space has a basis

We would like to prove that every vector space $\mathcal{V}$ has a basis.

The Grow algorithm and the Shrink algorithm each provides a way to prove this, but we are not there yet:

- The Grow-Algorithm Corollary implies that, if the Grow algorithm terminates, the set of vectors it has selected is a basis for the vector space $\mathcal{V}$.
  However, *we have not yet shown that it always terminates!*

- The Shrink-Algorithm Corollary implies that, if we can run the Shrink algorithm starting with a finite set of vectors that spans $\mathcal{V}$, upon termination it will have selected a basis for $\mathcal{V}$.
  However, *we have not yet shown that every vector space $\mathcal{V}$ is spanned by some finite set of vectors!*.

Computational problems involving finding a basis

Two natural ways to specify a vector space $\mathcal{V}$:

1. Specifying generators for $\mathcal{V}$.
2. Specifying a homogeneous linear system whose solution set is $\mathcal{V}$.

Two Fundamental Computational Problems:

Computational Problem: *Finding a basis of the vector space spanned by given vectors*

- *input:* a list $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ of vectors
- *output:* a list of vectors that form a basis for Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Computational Problem: *Finding a basis of the solution set of a homogeneous linear system*

- *input:* a list $[\mathbf{a}_1, \ldots, \mathbf{a}_n]$ of vectors
- *output:* a list of vectors that form a basis for the set of solutions to the system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_n \cdot \mathbf{x} = 0$