Quiz

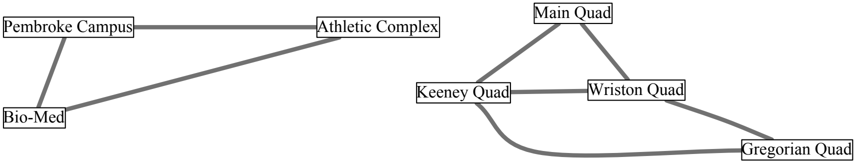
- What is the coordinate representation of $[1, 2, 3]$ in terms of the vectors $[1, 0, 0], [1, 1, 0], [1, 1, 1]$?
- Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ be vectors. Let $\mathbf{v}$ be a vector in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$. Suppose you are interested in finding the coordinate representation of $\mathbf{v}$ in terms of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$. How would you set up the problem using a matrix-vector equation?
- Suppose that $\mathbf{u}$ is the coordinate representation of some vector $\mathbf{v}$ in terms of $\mathbf{v}_1, \ldots, \mathbf{v}_5$. How would you find $\mathbf{v}$? Your answer should be formulated in terms of a matrix.

Activity: The image of a line under a linear transformation

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Consider a line L in $\mathbb{R}^n$ (not necessarily through the origin). What can you say about the image of L under f? (That is, the set of outputs corresponding to the elements of L as inputs.) Use algebra in an argument supporting your answer.

Hint: Recall our formulation of a line as the affine hull of a pair of vectors over $\mathbb{R}$.

Formulating *Minimum Spanning Forest* in linear algebra



The vector representing {`Keeney, Gregorian`},

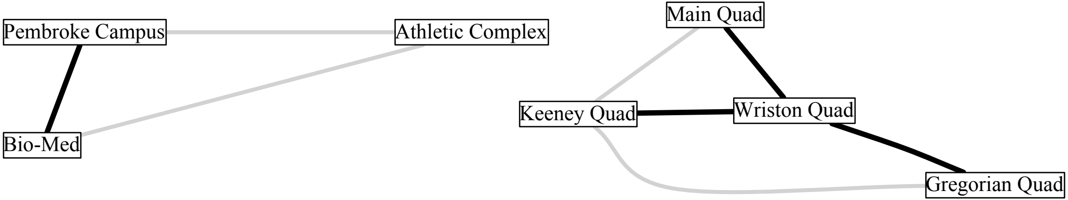
Pembroke	Athletic	Bio-Med	Main	Keeney	Wriston	Gregorian
				1		1

is the sum, for example, of the vectors representing {`Keeney, Main` }, {`Main, Wriston`}, and {`Wriston, Gregorian`} :

Pembroke	Athletic	Bio-Med	Main	Keeney	Wriston	Gregorian
			1	1		
			1		1	
					1	1

A vector with 1's in entries x and y is the sum of vectors corresponding to edges that form an x-to-y path in the graph.

Formulating *Minimum Spanning Forest* in linear algebra



A vector with 1's in entries x and y is the sum of vectors corresponding to edges that form an x-to-y path in the graph.

Example: The span of the vectors representing

{Pembroke, Bio-Med}, {Main, Wriston}, {Keeney, Wriston}, {Wriston, Gregorian }

- contains the vectors corresponding to

  {Main, Keeney}, {Keeney, Gregorian}, and {Main, Gregorian}

- but not the vectors corresponding to

  {Athletic, Bio-Med } or {Bio-Med, Main}.

Grow algorithms

```
def GROW(G)
    S := ∅
    consider the edges in increasing order
    for each edge e:
        if e's endpoints are not yet connected
            add e to S.
```

```
def GROW(𝒱)
    S = ∅
    repeat while possible:
        find a vector v in 𝒱 not in Span S,
        and put it in S.
```

- Considering edges e of G corresponds to considering vectors $\mathbf{v}$ in $\mathcal{V}$
- Testing if e's endpoints are not connected corresponds to testing if $\mathbf{v}$ is not in Span S.

The Grow algorithm for MSF is a specialization of the Grow algorithm for vectors.

Same for the Shrink algorithms.

Linear Dependence: The Superfluous-Vector Lemma

Grow and Shrink algorithms both test whether a vector is superfluous in spanning a vector space $\mathcal{V}$. Need a criterion for superfluity.

Superfluous-Vector Lemma: For any set S and any vector $\mathbf{v} \in S$, if $\mathbf{v}$ can be written as a linear combination of the other vectors in S then Span $(S - \{\mathbf{v}\})$ = Span S

Proof: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Suppose $\mathbf{v}_n = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_{n-1} \mathbf{v}_{n-1}$

To show: every vector in Span S is also in Span $(S - \{\mathbf{v}_n\})$.

Every vector $\mathbf{v}$ in Span S can be written as $\mathbf{v} = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \cdots \beta_n \mathbf{v}_n$

Substituting for $\mathbf{v}_n$, we obtain

$$\begin{aligned} \mathbf{v} &= \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \cdots + \beta_n \left(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_{n-1} \mathbf{v}_{n-1}\right) \\ &= (\beta_1 + \beta_n \alpha_1)\mathbf{v}_1 + (\beta_2 + \beta_n \alpha_2)\mathbf{v}_2 + \cdots + (\beta_{n-1} + \beta_n \alpha_{n-1})\mathbf{v}_{n-1} \end{aligned}$$

which shows that an arbitrary vector in Span S can be written as a linear combination of vectors in $S - \{\mathbf{v}_n\}$ and is therefore in Span $(S - \{\mathbf{v}_n\})$.

QED

Defining linear dependence

Definition: Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are *linearly dependent* if the zero vector can be written as a **nontrivial** linear combination of the vectors:

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

In this case, we refer to the linear combination as a *linear dependency* in $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

On the other hand, if the *only* linear combination that equals the zero vector is the trivial linear combination, we say $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly *in*dependent.

Example: The vectors $[1, 0, 0]$, $[0, 2, 0]$, and $[2, 4, 0]$ are linearly dependent, as shown by the following equation:

$$2\,[1, 0, 0] + 2\,[0, 2, 0] - 1\,[2, 4, 0] = [0, 0, 0]$$

Therefore:

$2\,[1, 0, 0] + 2\,[0, 2, 0] - 1\,[2, 4, 0]$ is a linear dependency in $[1, 0, 0]$, $[0, 2, 0]$, $[2, 4, 0]$.

Linear dependence

Example: The vectors $[1, 0, 0]$, $[0, 2, 0]$, and $[0, 0, 4]$ are linearly independent.
How do we know?
Easy since each vector has a nonzero entry where the others have zeroes.
Consider any linear combination

$$\alpha_1\,[1,0,0] + \alpha_2\,[0,2,0] + \alpha_3\,[0,0,4]$$

This equals $[\alpha_1, 2\alpha_2, 4\alpha_3]$

If this is the zero vector, it must be that $\alpha_1 = \alpha_2 = \alpha_3 = 0$
That is, the linear combination is trivial.

We have shown the only linear combination that equals the zero vector is the trivial linear combination.

Linear dependence in relation to other questions

How can we tell if vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent?

Definition: Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are *linearly dependent* if the zero vector can be written as a nontrivial linear combination $\mathbf{0} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$

By linear-combinations definition, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent iff there is a nonzero vector $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ such that $\begin{bmatrix} & & \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ & & \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \mathbf{0}$

Therefore, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent iff the null space of the matrix is nontrivial.

This shows that the question

How can we tell if vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent?

is the same as a question we asked earlier:

How can we tell if the null space of a matrix is trivial?

Linear dependence in relation to other questions

The question

How can we tell if vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent?

is the same as a question we asked earlier:

How can we tell if the null space of a matrix is trivial?

Recall: *solution set of a homogeneous linear system*

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= 0 \\ &\vdots \\ \mathbf{a}_m \cdot \mathbf{x} &= 0 \end{aligned}$$

is the null space of matrix $\left[\begin{array}{c} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{array}\right]$.

So question is same as:

How can we tell if the solution set of a homogeneous linear system is trivial?

Linear dependence in relation to other questions

The question

$$\boxed{\text{How can we tell if vectors } \mathbf{v}_1, \ldots, \mathbf{v}_n \text{ are linearly dependent?}}$$

is the same as a question we asked earlier:

$$\boxed{\text{How can we tell if the null space of a matrix is trivial?}}$$

is the same as :

$$\boxed{\text{How can we tell if the solution set of a homogeneous linear system is trivial?}}$$

Recall: If $\mathbf{u}_1$ is a solution to a linear system $\mathbf{a}_1 \cdot \mathbf{x} = \beta_1, \ldots, \mathbf{a}_m \cdot \mathbf{x} = \beta_m$ then

$$\{\text{solutions to linear system}\} = \{\mathbf{u}_1 + \mathbf{v} : \mathbf{v} \in \mathcal{V}\}$$

where $\mathcal{V} = \{$solutions to corresponding homogeneous linear system

$$\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0\}$$

Thus the question is the same as:

$$\boxed{\text{How can we tell if a solution } \mathbf{u}_1 \text{ to a linear system is the } \textit{only} \text{ solution?}}$$

Linear dependence and null space

The question

How can we tell if vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent?

is the same as:

How can we tell if the null space of a matrix is trivial?

is the same as:

How can we tell if the solution set of a homogeneous linear system is trivial?

is the same as:

How can we tell if a solution $\mathbf{u}_1$ to a linear system is the *only* solution?

Linear dependence

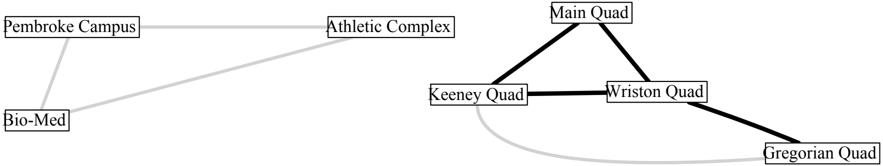
Answering these questions requires an algorithm.

Computational Problem: *Testing linear dependence*

- *input:* a list $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ of vectors
- *output:* DEPENDENDENT if the vectors are linearly dependent, and INDEPENDENT otherwise.

We'll see two algorithms later.

Linear dependence in *Minimum Spanning Forest*



We can get the zero vector by adding together vectors corresponding to edges that form a cycle: in such a sum, for each entry x, there are exactly two vectors having 1's in position x.

Example: the vectors corresponding to

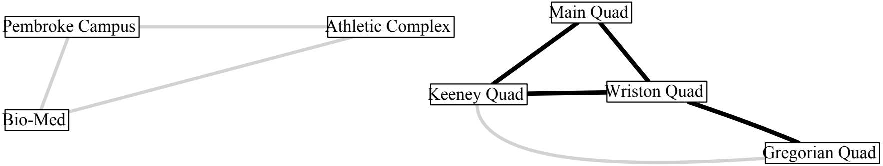
$$\{\texttt{Main, Wriston}\}, \{\texttt{Main, Keeney}\}, \{\texttt{Keeney, Wriston}\}$$

are as follows:

Pembroke	Athletic	Bio-Med	Main	Keeney	Wriston	Gregorian
			1	1		
				1	1	
			1		1	

The sum of these vectors is the zero vector.

Linear dependence in *Minimum Spanning Forest*



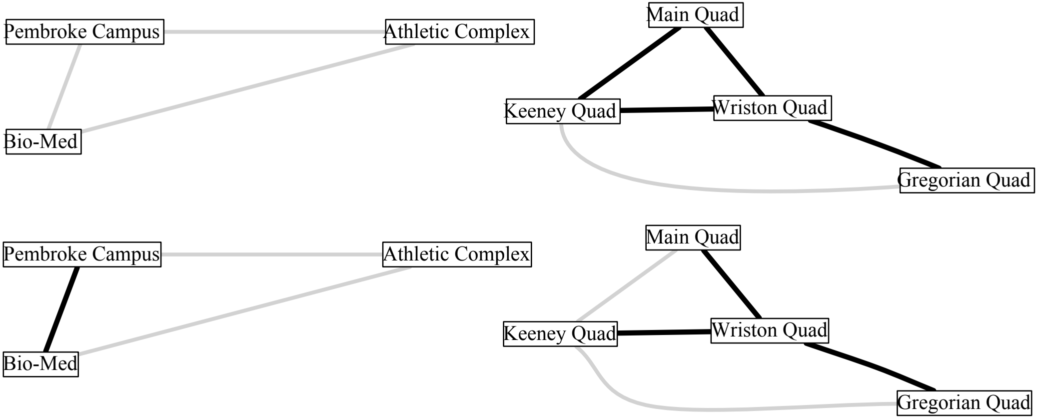
Sum of vectors corresponding to edges forming a cycle can make a zero vector.
Therefore if a subset of S form a cycle then S is linearly dependent.

Example: The vectors corresponding to
`{Main, Keeney}`, `{Main, Wriston}`, `{Keeney, Wriston }`, `{Wriston, Gregorian}`
are linearly dependent because these edges include a cycle.

The zero vector is equal to the nontrivial linear combination

			Pembroke	Athletic	Bio-Med	Main	Keeney	Wriston	Gregorian
	1	*				1	1		
+	1	*				1		1	
+	1	*					1	1	
+	0	*						1	1

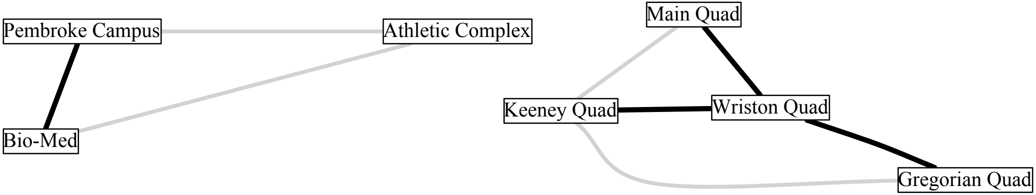
Linear dependence in *Minimum Spanning Forest*



If a subset of S form a cycle then S is linearly dependent.

On the other hand, if a set of edges contains no cycle (i.e. is a forest) then the corresponding set of vectors is linearly independent.

“Quiz”



Which edges are spanned?
Which sets are linearly dependent?

Properties of linear independence: hereditary

Lemma: If a finite set S of vectors is linearly dependent and S is a subset of T then T is linearly dependent.

In graphs, if a set S of edges includes a cycle then a superset of S also includes a cycle.

Properties of linear independence: hereditary

Lemma: If a finite set S of vectors is linearly dependent and S is a subset of T then T is linearly dependent.
Proof: If the zero vector can be written as a nontrivial linear combination of some vectors, it can be so written even if we allow some extra vectors to be in the linear combination because we can use zero coefficients on the extra vectors.

More formal proof: Write $S = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ and $T = \{\mathbf{s}_1, \ldots, \mathbf{s}_n, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_k\}$. Suppose S is linearly dependent. Then there are coefficients $\alpha_1, \ldots, \alpha_n$, not all zero, such that

$$\mathbf{0} = \alpha_1 \mathbf{s}_1 + \cdots + \alpha_n \mathbf{s}_n$$

Therefore

$$\mathbf{0} = \alpha_1 \mathbf{s}_1 + \cdots + \alpha_n \mathbf{s}_n + 0\, \boldsymbol{t}_1 + \cdots 0\, \boldsymbol{t}_k$$

which shows that the zero vector can be written as a nontrivial linear combination of the vectors of T,i.e. that T is linearly dependent.
QED

Properties of linear (in)dependence

Linear-Dependence Lemma Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors.

A vector $\mathbf{v}_i$ is in the span of the other vectors
if and only if
the zero vector can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$
in which the coefficient of $\mathbf{v}_i$ is nonzero.

In graphs, the Linear-Dependence Lemma states that an edge e is in the span of other edges if there is a cycle consisting of e and a subset of the other edges.

Properties of linear (in)dependence

Linear-Dependence Lemma Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors.

A vector $\mathbf{v}_i$ is in the span of the other vectors
if and only if
the zero vector can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$
in which the coefficient of $\mathbf{v}_i$ is nonzero.

Proof: *First direction:* Suppose $\mathbf{v}_i$ is in the span of the other vectors. That is, there exist coefficients $\alpha_1, \ldots, \alpha_{n-1}$ such that

$$\mathbf{v}_i = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_{i-1} \mathbf{v}_{i-1} + \alpha_{i+1} \mathbf{v}_{i+1} + \cdots \alpha_n \mathbf{v}_n$$

Moving $\mathbf{v}_i$ to the other side, we write

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \cdots + (-1)\mathbf{v}_i + \cdots + \alpha_n \mathbf{v}_n$$

which shows that the all-zero vector can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in which the coefficient of $\mathbf{v}_i$ is nonzero.

Properties of linear (in)dependence

Linear-Dependence Lemma Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors.

A vector $\mathbf{v}_i$ is in the span of the other vectors
if and only if
the zero vector can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$
in which the coefficient of $\mathbf{v}_i$ is nonzero.

Proof: *Other direction.* Suppose there are coefficients $\alpha_1, \ldots, \alpha_n$ such that

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_i \mathbf{v}_i + \cdots + \alpha_n \mathbf{v}_n$$

and such that $\alpha_i \neq 0$.

Dividing both sides by α_i yields

$$\mathbf{0} = (\alpha_1/\alpha_i)\, \mathbf{v}_1 + (\alpha_2/\alpha_i)\, \mathbf{v}_2 + \cdots + \mathbf{v}_i + \cdots + (\alpha_n/\alpha_i)\, \mathbf{v}_n$$

Moving every term from right to left except $\mathbf{v}_i$ yields

$$-(\alpha_1/\alpha_i)\, \mathbf{v}_1 - (\alpha_2/\alpha_i)\, \mathbf{v}_2 - \cdots - (\alpha_n/\alpha_i)\, \mathbf{v}_n = \mathbf{v}_i$$

QED