Quiz

Define the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ as follows:

$$\mathbf{a}_1 = \begin{array}{ccc} \# & \$ & \% \\ \hline 2 & 1 & -1 \end{array} \text{ and } \mathbf{a}_2 = \begin{array}{ccc} \# & \$ & \% \\ \hline -1 & 4 & 3 \end{array}$$

Let A be the matrix whose column-dict representation is $\{'@' : \mathbf{a}_1, '\&' : \mathbf{a}_2\}$. Compute A times the vector $\begin{array}{cc} @ & \& \\ \hline 2 & 4 \end{array}$

Let B be the matrix whose row-dict representation is $\{'@' : \mathbf{a}_1, '\&' : \mathbf{a}_2\}$. Compute B times the vector $\begin{array}{ccc} \# & \$ & \% \\ \hline -1 & 0 & 3 \end{array}$.

Matrices as vectors

Soon we study true matrix operations. But first....

A matrix can be interpreted as a vector:

- an $R \times S$ matrix is a function from $R \times S$ to $\mathbb{F}$,
- so it can be interpreted as an $R \times S$-vector:
  - *scalar-vector multiplication*
  - *vector addition*
- Our full implementation of `Mat` class will include these operations.

Transpose

Transpose swaps rows and columns.

```
         @  #  ?
      ---------
a   |    2  1  3
b   |   20 10 30
```

→

```
         a  b
      ------
@   |    2 20
#   |    1 10
?   |    3 30
```

Transpose (and Quiz)

Quiz: Write `transpose(M)`

Answer:

```
def transpose(M):
    return Mat((M.D[1], M.D[0]), {(q,p):v for (p,q),v in M.f.items()})
```

Computing sparse matrix-vector product

To compute matrix-vector or vector-matrix product,

- could use dot-product or linear-combinations definition.
- However, using those definitions, it's not easy to exploit sparsity in the matrix.

"Ordinary" Definition of Matrix-Vector Multiplication: If M is an $R \times C$ matrix and $\mathbf{u}$ is a C-vector then $M * \mathbf{u}$ is the R-vector $\mathbf{v}$ such that, for each $r \in R$,

$$v[r] = \sum_{c \in C} M[r, c] u[c]$$

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Obvious method:

```
1 for i in R:
2     v[i] := Σ_{j∈C} M[i,j]u[j]
```

But this doesn't exploit sparsity!

Idea:

- Initialize output vector $\mathbf{v}$ to zero vector.
- Iterate over nonzero entries of M, adding terms according to ordinary definition.

```
1 initialize v to zero vector
2 for each pair (i,j) in sparse representation,
3     v[i] = v[i] + M[i,j]u[j]
```

Linear systems using matrices

Recall: A linear system is a system of linear equations

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= \beta_1 \\ &\vdots \\ \mathbf{a}_m \cdot \mathbf{x} &= \beta_m \end{aligned}$$

Key idea: Write linear system as a matrix-vector equation.

$$\begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix} \mathbf{x} = \mathbf{b} \; \sout{[\beta_1, \ldots, \beta_m]} [\beta_1, \ldots, \beta_m]$$

We have many questions about linear systems:

- How to find a solution?
- How many solutions?
  - Does a solution even exist?
  - When does only one solution exist?
- What to do if there are no solutions?
- For which right-hand sides $\beta_1, \ldots, \beta_m$ does a solution exist?

Matrices will help us answer these questions and use linear systems in other applications.

First advantage of a matrix: Can interpret a row matrix as a column matrix:

$$\left[\begin{array}{c|c|c} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{array} \right] \mathbf{x} = \mathbf{b}$$

Interpret this matrix-vector equation as: With what coefficients can $\mathbf{b}$ be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$?

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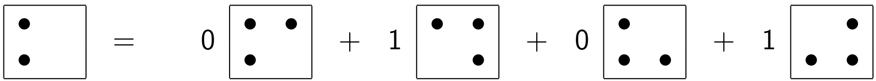
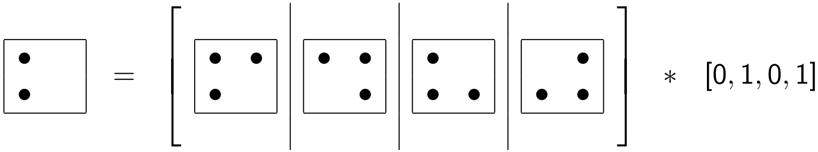
A solution to a *Lights Out* configuration is a linear combination of "button vectors."

For example, the linear combination



can be written as



Solving an instance of *Lights Out* ⇒ Solving a matrix-vector equation

Linear systems using matrices

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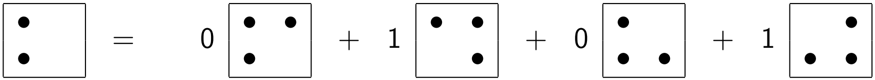

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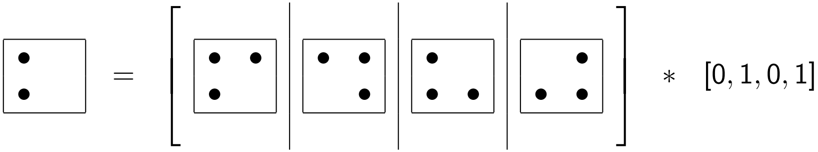
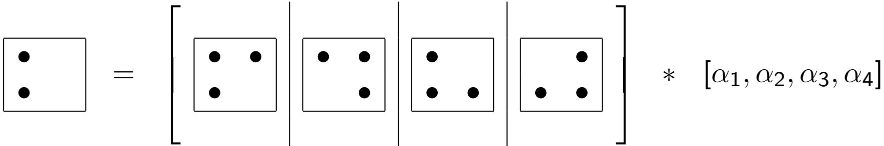


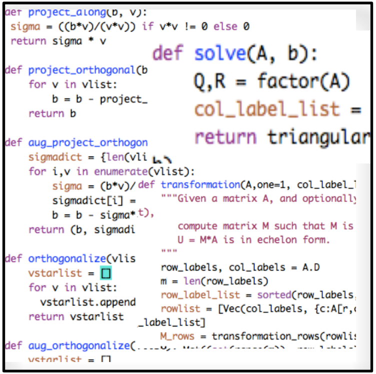
Solving an instance of *Lights Out* $\Rightarrow$ Solving a matrix-vector equation



The `solver` module

We provide a module `solver` that defines a procedure `solve(A, b)` that tries to find a solution to the matrix-vector equation $A * \mathbf{x} = \mathbf{b}$

Currently `solve(A, b)` is a black box



but we will learn how to code it in the coming weeks.

Let's use it to solve this *Lights Out* instance...

The two fundamental spaces associated with a matrix

Want to study linear systems, equivalently *matrix-vector equations* $A\mathbf{x} = \mathbf{b}$

For which right-hand side vectors $\mathbf{b}$ does $A\mathbf{x} = \mathbf{b}$ have a solution?

An $R \times C$ matrix A corresponds to the function $f : \mathbb{F}^C \longrightarrow \mathbb{F}^R$ defined by $f(\mathbf{x}) = A\mathbf{x}$

The system $A\mathbf{x} = \mathbf{b}$ has a solution *if and only if* there is some vector $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{b}$.

Thus the system has a solution *if and only if* there is some vector $\mathbf{v}$ in $\mathbb{F}^C$ such that $f(\mathbf{v}) = \mathbf{b}$.

Thus the system has a solution *if and only if* $\mathbf{b}$ is in the image of the function f.

(Remember that the *image* of a function is the set of all possible outputs.)

Question: How can we characterize the image of the function $f(\mathbf{x}) = A\mathbf{x}$?

Answer: Use the linear-combinations interpretation.

For any input $\mathbf{x}$, the output is the linear combination of the columns of A where coefficients are the entries of $\mathbf{x}$.

The set of **all** outputs is thus the set of all linear combinations of the columns of A.

Also known as the *span* of the columns of A

We call this the column space of A. Written Col A.

We know it is a vector space.

The two fundamental spaces associated with a matrix

Recall: We are interested in solution set of corresponding homogeneous linear system

$$\begin{bmatrix} & A & \end{bmatrix} \mathbf{x} = \mathbf{0}$$

We know this is a vector space.
In context of matrices, we call it the null space of the matrix A. Written Null A

Summary: The two most important spaces associated with a matrix A are

- Col A
- Null A

Solution set of a matrix-vector equation

Proposition: If $\mathbf{u}_1$ is a solution to $A * \mathbf{x} = \mathbf{b}$ then solution set is $\mathbf{u}_1 + \mathcal{V}$ where $\mathcal{V} = \text{Null } A$

- If $\mathcal{V}$ is a trivial vector space then $\mathbf{u}_1$ is the only solution.
- If $\mathcal{V}$ is not trivial then $\mathbf{u}_1$ is *not* the only solution.

Corollary: $A * \mathbf{x} = \mathbf{b}$ has at most one solution iff Null A is a trivial vector space.

Question: How can we tell if the null space of a matrix is trivial?

Answer comes later...

Matrix-matrix multiplication

If

- A is a $R \times S$ matrix, and
- B is a $S \times T$ matrix

then it is legal to multiply A times B.

- In Mathese, written AB
- In our `Mat` class, written `A*B`

AB is different from BA.

In fact, one product might be legal while the other is illegal.

Matrix-matrix multiplication: matrix-vector definition

Matrix-vector definition of matrix-matrix multiplication:

For each column-label s of B,

$$\text{column } s \text{ of } AB = A * (\text{column } s \text{ of } B)$$

Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $B =$ matrix with columns $[4, 3]$, $[2, 1]$, and $[0, -1]$

$$B = \left[\begin{array}{c|c|c} 4 & 2 & 0 \\ 3 & 1 & -1 \end{array}\right]$$

AB is the matrix with column $i = A * ($ column i of $B)$

$A * [4, 3] = [10, -1]$ $\quad$ $A * [2, 1] = [4, -1]$ $\quad$ $A * [0, -1] = [-2, -1]$

$$AB = \left[\begin{array}{c|c|c} 10 & 4 & -2 \\ -1 & -1 & -1 \end{array}\right]$$

Matrix-matrix multiplication: Dot-product definition

Combine

- *matrix-vector* definition of matrix-matrix multiplication, and
- *dot-product* definition of matrix-vector multiplication

to get...

Dot-product definition of matrix-matrix multiplication:
Entry rc of AB is the dot-product of row r of A with column c of B.

Example:

$$\left[\begin{array}{ccc} 1 & 0 & 2 \\ \hline 3 & 1 & 0 \\ \hline 2 & 0 & 1 \end{array}\right] \left[\begin{array}{c|c} 2 & 1 \\ 5 & 0 \\ 1 & 3 \end{array}\right] = \left[\begin{array}{cc} [1,0,2]\cdot[2,5,1] & [1,0,2]\cdot[1,0,3] \\ [3,1,0]\cdot[2,5,1] & [3,1,0]\cdot[1,0,3] \\ [2,0,1]\cdot[2,5,1] & [2,0,1]\cdot[1,0,3] \end{array}\right] = \left[\begin{array}{cc} 4 & 7 \\ 11 & 3 \\ 5 & 5 \end{array}\right]$$

Matrix-matrix multiplication: transpose

$$(AB)^T = B^T A^T$$

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 19 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 5 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 19 \\ 4 & 8 \end{bmatrix}$$

You might think "$(AB)^T = A^T B^T$" but this is **false**.

In fact, doesn't even make sense!

- For AB to be legal, A's column labels $=$ B's row labels.
- For $A^T B^T$ to be legal, A's row labels $=$ B's column labels.

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$ is legal but $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ is not.

Matrix-matrix multiplication: Column vectors

Multiplying a matrix A by a one-column matrix B

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \\ \mathbf{b} \\ \\ \end{bmatrix}$$

By matrix-vector definition of matrix-matrix multiplication, result is matrix with one column: $A * \mathbf{b}$

This shows that matrix-vector multiplication is subsumed by matrix-matrix multiplication.

Convention: Interpret a vector $\mathbf{b}$ as a one-column matrix ("column vector")

- Write vector $[1, 2, 3]$ as $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
- Write $A * [1, 2, 3]$ as $\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ or $A\,\mathbf{b}$