Quiz

Let $\mathbf{a}_1 = [1, 0, -1], \mathbf{a}_2 = [2, 1, 0], \mathbf{a}_3 = [10, 1, 2], \mathbf{a}_4 = [0, 0, 1]$. Compute the row-matrix-by-vector product

$$\begin{bmatrix} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \\ \hline \mathbf{a}_3 \\ \hline \mathbf{a}_4 \end{bmatrix} \text{ times } [7, 6, 5]$$

Let $\mathbf{v}_1 = [2, 1, 0, 1], \mathbf{v}_2 = [4, -1, 0, 2], \mathbf{v}_3 = [0, 1, 0, 1]$. Compute the column-matrix-by-vector product

$$\left[\begin{array}{c|c|c} & & \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ & & \end{array}\right] \text{ times } [2, -1, 1]$$

[3] The Matrix

Neo: What is the Matrix?
Trinity: The answer is out there, Neo, and it's looking for you, and it will find you if you want it to. *The Matrix*, 1999

Two views of same process

Row-matrix-by-vector multiplication and column-matrix-by-vector multiplication are same!

$$\left[\begin{array}{c|c|c} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} & \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} & \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \end{array}\right] \text{ times } [x_1, x_2, x_3]$$

$$= x_1 \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} + x_2 \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} + x_3 \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

$$= \begin{bmatrix} x_1 a_1 \\ x_1 a_2 \\ \vdots \\ x_1 a_m \end{bmatrix} + \begin{bmatrix} x_2 b_1 \\ x_2 b_2 \\ \vdots \\ x_2 b_m \end{bmatrix} + \begin{bmatrix} x_3 c_1 \\ x_3 c_2 \\ \vdots \\ x_3 c_m \end{bmatrix}$$

Two views of same process

Row-matrix-by-vector multiplication and column-matrix-by-vector multiplication are same!

$$
\begin{aligned}
&= x_1 \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} + x_2 \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} + x_3 \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \\
&= \begin{bmatrix} x_1 a_1 \\ x_1 a_2 \\ \vdots \\ x_1 a_m \end{bmatrix} + \begin{bmatrix} x_2 b_1 \\ x_2 b_2 \\ \vdots \\ x_2 b_m \end{bmatrix} + \begin{bmatrix} x_3 c_1 \\ x_3 c_2 \\ \vdots \\ x_3 c_m \end{bmatrix} \\
&= \begin{bmatrix} x_1 a_1 + x_2 b_1 + x_3 c_1 \\ x1 a_2 + x_2 b_2 + x_3 c_2 \\ \vdots \\ x_1 a_m + x_2 b_m + x_3 c_m \end{bmatrix}
\end{aligned}
$$

Two views of same process

Row-matrix-by-vector multiplication and column-matrix-by-vector multiplication are same!

$$= \begin{bmatrix} x_1 a_1 \\ x_1 a_2 \\ \vdots \\ x_1 a_m \end{bmatrix} + \begin{bmatrix} x_2 b_1 \\ x_2 b_2 \\ \vdots \\ x_2 b_m \end{bmatrix} + \begin{bmatrix} x_3 c_1 \\ x_3 c_2 \\ \vdots \\ x_3 c_m \end{bmatrix}$$

$$= \begin{bmatrix} x_1 a_1 + x_2 b_1 + x_3 c_1 \\ x1 a_2 + x_2 b_2 + x_3 c_2 \\ \vdots \\ x_1 a_m + x_2 b_m + x_3 c_m \end{bmatrix}$$

Two views of same process

$$= \begin{bmatrix} x_1 a_1 + x_2 b_1 + x_3 c_1 \\ x1 a_2 + x_2 b_2 + x_3 c_2 \\ \vdots \\ x_1 a_m + x_2 b_m + x_3 c_m \end{bmatrix}$$

$$= \begin{bmatrix} (a_1, b_1, c_1) \cdot (x_1, x_2, x_3) \\ (a_2, b_2, c_2) \cdot (x_1, x_2, x_3) \\ \vdots \\ (a_m, b_m, c_m) \cdot (x_1, x_2, x_3) \end{bmatrix}$$

$$= \left[\begin{array}{c} [a_1, b_1, c_1] \\ \hline [a_2, b_2, c_2] \\ \hline \vdots \\ \hline [a_m, b_m, c_m] \end{array}\right] \text{ times } [x_1, x_2, x_3]$$

Two views of same process

We showed $\left[\begin{array}{c|c|c} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} & \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} & \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \end{array}\right]$ times $[x_1, x_2, x_3]$

$$= \left[\begin{array}{c} [a_1, b_1, c_1] \\ \hline [a_2, b_2, c_2] \\ \hline \vdots \\ \hline [a_m, b_m, c_m] \end{array}\right] \text{ times } [x_1, x_2, x_3]$$

so $\left[\begin{array}{c|c|c} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} & \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} & \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \end{array}\right]$ is same as $\left[\begin{array}{c} [a_1, b_1, c_1] \\ \hline [a_2, b_2, c_2] \\ \hline \vdots \\ \hline [a_m, b_m, c_m] \end{array}\right]$

Row matrix and column matrix are same

Example: $\left[\begin{array}{c|c|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array}\right]$ same as $\left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \end{array}\right]$

Same underlying math'l object, different representations

- *column-list* representation
- *row-list* representation

of a MATRIX

One operation, matrix-vector multiplication,
with two interpretations:

- dot-product interpretation: output vector entries are dot-products of rows with input vector
- linear-combinations interpretation: output vector is linear combination of columns where coeff's are input vector entries

You must memorize which is which.

The Matrix

Traditional notion of a matrix: two-dimensional array.

$$\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix}$$

- Two rows: $[1, 2, 3]$ and $[10, 20, 30]$.
- Three columns: $[1, 10]$, $[2, 20]$, and $[3, 30]$.
- A 2×3 matrix.

For a matrix A, the i, j element of A

- is the element in row i, column j
- is traditionally written $A_{i,j}$
- but we will use $A[i, j]$

List of row-lists, list of column-lists

- One obvious Python representation for a matrix: a list of row-lists:
  $\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix}$ represented by `[[1,2,3],[10,20,30]]`.
- Another: a list of column-lists:
  $\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix}$ represented by `[[1,10],[2,20],[3,30]]`.

List of row-lists, list of column-lists

Ungraded "Quiz": Write a nested comprehension whose value is list-of-*row*-list representation of a 3×4 matrix all of whose elements are zero:

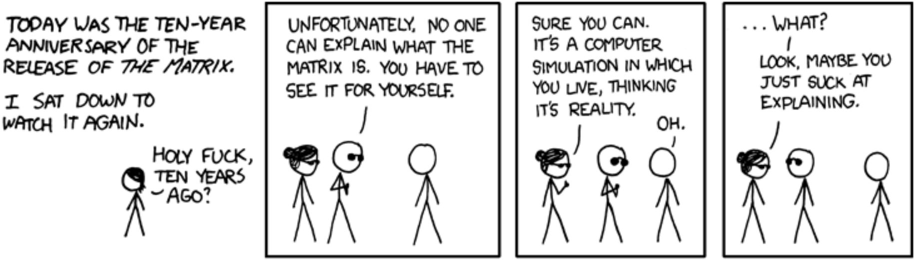
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hint: first write a comprehension for a typical row, then use that expression in a comprehension for the list of lists.

Answer:

```
>>> [[0 for j in range(4)] for i in range(3)]
[[0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0]]
```

The matrix revealed



The Matrix Revisited (excerpt) `http://xkcd.com/566/`

Definition: For finite sets R and C, an $R \times C$ *matrix over* $\mathbb{F}$ is a function from $R \times C$ to $\mathbb{F}$.

	@	#	?
a	1	2	3
b	10	20	30

- $R = \{\texttt{a}, \texttt{b}\}$ and $C = \{\texttt{@}, \texttt{\#}, \texttt{?}\}$.
- R is set of row labels
- C is set of column labels

In Python, the function is represented by a dictionary:

```
{('a','@'):1, ('a','#'):2, ('a', '?'):3,
 ('b', '@'):10, ('b',  '#'):20, ('b','?'):30}
```

Rows, columns, and entries

	@	#	?
a	1	2	3
b	10	20	30

Rows and columns are vectors, e.g.

- Row `'a'` is the vector `Vec({'@', '#', '?'}, {'@':1, '#':2, '?':3})`
- Column `'#'` is the vector `Vec({'a','b'}, {'a':2, 'b':20})`

Dict-of-rows/dict-of-columns representations

	@	#	?
a	1	2	3
b	10	20	30

One representation: *dictionary of rows:*

```
{'a': Vec({'#', '@', '?'}, {'@':1, '#':2, '?':3}),
 'b': Vec({'#', '@', '?'}, {'@':10, '#':20, '?':30})}
```

Another representation: *dictionary of columns:*

```
{'@': Vec({'a','b'}, {'a':1, 'b':10}),
 '#': Vec({'a','b'}, {'a':2, 'b':20}),
 '?': Vec({'a','b'}, {'a':3, 'b':30})}
```

Our Python implementation

	@	#	?
a	1	2	3
b	10	20	30

```
>>> M=Mat(({'a','b'}, {'@', '#', '?'}),
          {('a','@'):1, ('a','#'):2,('a','?'):3,
           ('b','@'):10, ('b','#'):20, ('b','?'):30})
```

A class with two fields:

- D, a *pair* (R, C) of sets.
- f, a dictionary representing a function that maps pairs $(r, c) \in R \times C$ to field elements.

```
class Mat:
    def __init__(self, labels, function):
        self.D = labels
        self.f = function
```

We will later add lots of matrix operations to this class.

Example: For a Mat M, $M[r, c]$ is the entry in row r, column c.

Identity matrix

For any domain D, there is a matrix that represents the D-to-D identity function $f(\mathbf{x}) = \mathbf{x}$

```
       a b c
      -------
 a  |  1 0 0
 b  |  0 1 0
 c  |  0 0 1
```

Definition: $D \times D$ *identity matrix* is the matrix $\mathbb{1}_D$ such that $\mathbb{1}_D[k, k] = 1$ for all $k \in D$ and zero elsewhere.

Usually we omit the subscript when D is clear from the context.
Often letter `I` (for "identity") is used instead of $\mathbb{1}$

```
Mat(({'a','b','c'},{'a','b','c'}),{('a','a'):1,('b','b'):1,('c','c'):1})
```

Quiz: Write procedure `identity`(D) that returns the $D \times D$ identity matrix over $\mathbb{R}$ represented as an instance of `Mat`.

Answer: `def identity(D): return Mat((D,D), (k,k):1 for k in D)`

Converting between representations

Converting an instance of `Mat` to a column-dictionary representation:

	@	#	?
a	1	2	3
b	10	20	30

```
Mat(({'a','b'}, {'@', '#', '?'}), {('a','@'):1, ('a','#'):2,
          ('a','?'):3, ('b','@'):10, ('b','#'):20, ('b','?'):30})
```

$$\Rightarrow$$

```
{'@': Vec({'a','b'}, {'a':1, 'b':10}),
 '#': Vec({'a','b'}, {'a':2, 'b':20}),
 '?': Vec({'a','b'}, {'a':3, 'b':30})}
```

Quiz: Write the procedure `mat2coldict(A)` that, given an instance of `Mat`, returns the column-dictionary representation of the same matrix.

Answer:

```
def mat2coldict(A):
 return {c:Vec(A.D[0],{r:A[r,c] for r in A.D[0]}) for c in A.D[1]}
```

Module `matutil`

We provide a module, `matutil`, that defines several conversion routines:

- `mat2coldict(A)`: from a `Mat` to a dictionary of columns represented as `Vecs`)
- `mat2rowdict(A)`: from a `Mat` to a dictionary of rows represented as `Vecs`
- `coldict2mat(coldict)` from a dictionary of columns (or a list of columns) to a `Mat`
- `rowdict2mat(rowdict)`: from a dictionary of rows (or a list of rows) to a `Mat`
- `listlist2mat(L)`: from a list of list of field elements to a `Mat` the inner lists turn into rows

and also:

- `identity(D, one)`: produce a `Mat` representing the $D \times D$ identity matrix

The Mat class

We gave the definition of a rudimentary matrix class:

```
class Mat:
    def __init__(self,
          labels, function):
           self.D = labels
           self.f = function
```

The more elaborate class definition allows for more concise vector code, e.g.

```
>>> M['a', 'B'] = 1.0
>>> b = M*v
>>> B = M*A
>>> print(B)
```

More elaborate version of this class definition allows operator overloading for element access, matrix-vector multiplication, etc.

operation	syntax
Matrix addition and subtraction	`A+B` and `A-B`
Matrix negative	`-A`
Scalar-matrix multiplication	`alpha*A`
Matrix equality test	`A == B`
Matrix transpose	`A.transpose()`
Getting a matrix entry	`A[r,c]`
Matrix-vector multiplication	`A*v`
Matrix-matrix multiplication	`A*B`

You will code this class starting from a template we provide.

Using Mat

You will write the bodies of named procedures such as `setitem(M, k, val)` and `matrix_vector_mul(M, v)` and `transpose(M)`.

In using Mats in other code, you must use operators and methods instead of named procedures, e.g.

```
>>> M['a', 'b'] = 1.0
>>> v = M*u
>>> b_parallel =
      Q*Q.transpose()*b
```

instead of

```
>>> setitem(M, ('a','B'), 1.0)
>>> v = matrix_vector_mul(M, u)
>>> b_parallel =
    matrix_vector_mul(matrix_matrix_mul(Q,
                                        transpose(Q)), b)
```

In code outside the `mat` module that uses `Mat`, you will import just `Mat` from the `mat` module:

```
from mat import Mat
```

so named procedures will not be imported into the namespace.

In short: Use the operators [ ], +, *, - and the method `.transpose()` when working with Mats

Assertions in Mat

For each procedure you write, we will provide the stub of the procedure, e.g. for `matrix_vector_mul(M, v)`, we provide the stub

```
def matrix_vector_mul(M, v):
    "Returns the product of matrix M and vector v"
    assert M.D[1] == v.D
    pass
```

You are supposed to replace the pass statement with code for the procedure.

The first line in the body is a documentation string.

The second line is an assertion. It asserts that the second element of the pair `M.D`, the set of column-labels of `M`, must be equal to the domain of the vector `v`. If the procedure is called with arguments that violate this, Python reports an error.

The assertion is there to remind us of a rule about matrix-vector multiplication.

Please keep the assertions in your `mat` code while using it for this course.

Testing Mat with doctests

Because you will use Mat a lot, making sure your implementation is correct will save you from lots of pain later.

Akin to Vec, we have provided doctests

```
def getitem(M, k):
    """
    Returns value of entry k in M
    >>> M = Mat(({1,3,5}, {'a'}),
             {(1,'a'):4, (5,'a'): 2})
    >>> M[1,'a']
    4
    """
    pass
```

You can test using copy-paste:

```
>>> from vec import Mat
>>> M = Mat(({1,3,5}, {'a'}), ...
>>> M[1,'a']
4
```

You can also run all the tests at once from the console (outside the Python interpreter) using the following command:

```
python3 -m doctest mat.py
```