Quiz

- Define *linear combination* and give two examples using the 3-vectors $\mathbf{v}_1 = [1, 1, 0], \mathbf{v}_2 = [3, 1, 1]$ over $\mathbb{R}$.
- Define *span* of $\{\mathbf{v}_1, \mathbf{v}_2\}$.
- What does it mean for $\mathbf{v}_1, \mathbf{v}_2$ to be *generators* of a set $\mathcal{V}$ of vectors?

Geometry of sets of vectors: span of vectors over $\mathbb{R}$

Span of a single nonzero vector **v**:

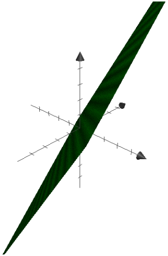
$$\text{Span } \{\mathbf{v}\} = \{\alpha\, \mathbf{v} \ : \ \alpha \in \mathbb{R}\}$$

This is the line through the origin and **v**. *One-dimensional*

Span of the empty set:just the origin. *Zero-dimensional*

Span $\{[1, 2], [3, 4]\}$: all points in the plane. *Two-dimensional*

Span of two 3-vectors? Span $\{[1, 0, 1.65], [0, 1, 1]\}$ is a plane in three dimensions:



Two-dimensional

Geometry of sets of vectors: span of vectors over $\mathbb{R}$

Is the span of k vectors always k-dimensional?

No.

- Span $\{[0, 0]\}$ is 0-dimensional.
- Span $\{[1, 3], [2, 6]\}$ is 1-dimensional.
- Span $\{[1, 0, 0], [0, 1, 0], [1, 1, 0]\}$ is 2-dimensional.

Fundamental Question: How can we predict the dimensionality of the span of some vectors?

Geometry of sets of vectors: span of vectors over $\mathbb{R}$

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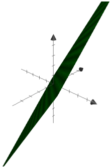
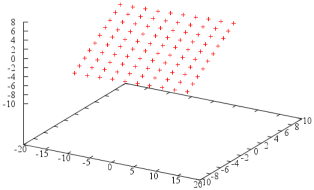


Two-dimensional

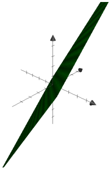
Useful for plotting the plane



$$\{\alpha\,[1, 0.1.65] + \beta\,[0, 1, 1] \quad : \quad \alpha \in \{-5, -4, \ldots, 3, 4\}, \; \beta \in \{-5, -4, \ldots, 3, 4\}\}$$

Geometry of sets of vectors: span of vectors over $\mathbb{R}$

Span of two 3-vectors? Span $\{[1, 0, 1.65], [0, 1, 1]\}$ is a plane in three dimensions



Perhaps a more familiar way to specify a plane:

$$\{(x, y, z) \ : \ ax + by + cz = 0\}$$

Using dot-product, we could rewrite as

$$\{[x, y, z] \ : \ [a, b, c] \cdot [x, y, z] = 0\}$$

Set of vectors satisfying a linear equation with right-hand side *zero*.

We can similarly specify a line in three dimensions:

$$\{[x, y, z] \ : \ \mathbf{a}_1 \cdot [x, y, z] = 0, \mathbf{a}_2 \cdot [x, y, z] = 0\}$$

Two ways to represent a geometric object (line, plane, etc.) containing the origin:

- Span of some vectors
- Solution set of some system of linear equations with zero right-hand sides

Geometry of sets of vectors: Two representations

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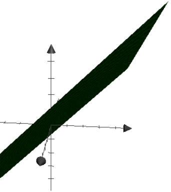
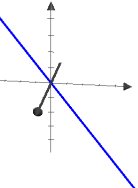


Span $\{[4, -1, 1], [0, 1, 1]\}$ $\{[x, y, z] \; : \; [1, 2, -2] \cdot [x, y, z] = 0\}$



Span $\{[1, 2, -2]\}$

$\{[x, y, z] \; :$
$[4, -1, 1] \cdot [x, y, z] = 0,$
$[0, 1, 1] \cdot [x, y, z] = 0\}$

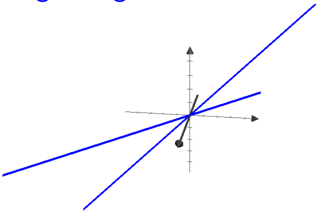
Geometry of sets of vectors: Two representations

Two ways to represent a geometric object (line, plane, etc.) containing the origin:

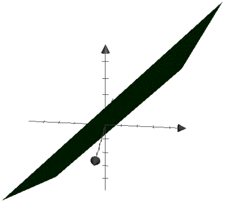
- Span of some vectors
- Solution set of some system of linear equations with zero right-hand sides

Each representation has its uses. Finding the plane containing two given lines:

- First line is Span $\{[4,-1,1]\}$.
- Second line is Span $\{[0,1,1]\}$.



- The plane containing these two lines is Span $\{[4,-1,1],[0,1,1]\}$



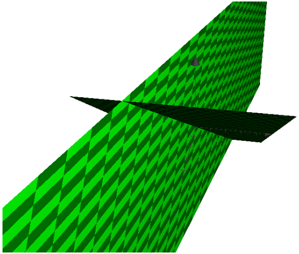
Geometry of sets of vectors: Two representations

Two ways to represent a geometric object (line, plane, etc.) containing the origin:

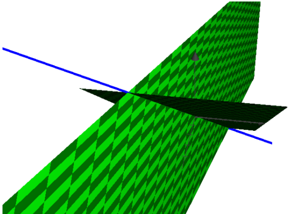
- Span of some vectors
- Solution set of some system of linear equations with zero right-hand sides

Each representation has its uses. Finding the intersection of two given planes:

- First plane is $\{[x, y, z] \; : \; [4, -1, 1] \cdot [x, y, z] = 0\}$.
- Second plane is $\{[x, y, z] \; : \; [0, 1, 1] \cdot [x, y, z] = 0\}$.



- The intersection is $\{[x, y, z] \; : \; [4, -1, 1] \cdot [x, y, z] = 0, [0, 1, 1] \cdot [x, y, z] = 0\}$



Two representations: What's common?

Subset of $\mathbb{F}^D$ that satisfies three properties:

Property V1 Subset contains the zero vector $\mathbf{0}$

Property V2 If subset contains $\mathbf{v}$ then it contains $\alpha\,\mathbf{v}$ for every scalar α

Property V3 If subset contains $\mathbf{u}$ and $\mathbf{v}$ then it contains $\mathbf{u}+\mathbf{v}$

Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ satisfies

- Property V1 because $0\,\mathbf{v}_1 + \cdots + 0\,\mathbf{v}_n$
- Property V2 because
$$\text{if } \mathbf{v} = \beta_1\,\mathbf{v}_1 + \cdots + \beta_n\,\mathbf{v}_n \text{ then } \alpha\,\mathbf{v} = \alpha\,\beta_1\mathbf{v}_1 + \cdots + \alpha\,\beta_n\,\mathbf{v}_n$$
- Property V3 because
$$\begin{aligned} \text{if } \mathbf{u} &= \alpha_1\,\mathbf{v}_1 + \cdots + \alpha_n\,\mathbf{v}_n \\ \text{and } \mathbf{v} &= \beta_1\,\mathbf{v}_1 + \cdots + \beta_n\,\mathbf{v}_n \\ \text{then } \mathbf{u}+\mathbf{v} &= (\alpha_1+\beta_1)\mathbf{v}_1 + \cdots + (\alpha_n+\beta_n)\,\mathbf{v}_n \end{aligned}$$

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Solution set $\{\mathbf{x} \; : \; \mathbf{a}_1 \cdot \mathbf{x} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{x} = 0\}$ satisfies

- Property V1 because

$$\mathbf{a}_1 \cdot \mathbf{0} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{0} = 0$$

- Property V2 because

$$\text{if } \mathbf{a}_1 \cdot \mathbf{v} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{v} = 0$$
$$\text{then } \mathbf{a}_1 \cdot (\alpha\,\mathbf{v}) = \alpha\,(\mathbf{a}_1 \cdot \mathbf{v}) = 0, \cdots, \mathbf{a}_m \cdot (\alpha\,\mathbf{v}) = \alpha\,(\mathbf{a}_m \cdot \mathbf{v}) = 0$$

- Property V3 because

$$\text{if } \mathbf{a}_1 \cdot \mathbf{u} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{u} = 0$$
$$\text{and } \mathbf{a}_1 \cdot \mathbf{v} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{v} = 0$$
$$\text{then } \mathbf{a}_1 \cdot (\mathbf{u}+\mathbf{v}) = \mathbf{a}_1 \cdot \mathbf{u} + \mathbf{a}_1 \cdot \mathbf{v} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot (\mathbf{u}+\mathbf{v}) = \mathbf{a}_m \cdot \mathbf{u} + \mathbf{a}_m \cdot \mathbf{v} = 0$$

Two representations: What's common?

Subset of $\mathbb{F}^D$ that satisfies three properties:

Property V1 Subset contains the zero vector $\mathbf{0}$

Property V2 If subset contains $\mathbf{v}$ then it contains $\alpha\,\mathbf{v}$ for every scalar α

Property V3 If subset contains $\mathbf{u}$ and $\mathbf{v}$ then it contains $\mathbf{u} + \mathbf{v}$

Any subset $\mathcal{V}$ of $\mathbb{F}^D$ satisfying the three properties is called a *subspace* of $\mathbb{F}^D$.

Example: Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{x} \ : \ \mathbf{a}_1 \cdot \mathbf{x} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{x} = 0\}$ are *subspaces* of $\mathbb{R}^D$

Possibly profound fact we will learn later: Every subspace of $\mathbb{F}^D$

- can be written in the form Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$
- can be written in the form $\{\mathbf{x} \ : \ \mathbf{a}_1 \cdot \mathbf{x} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{x} = 0\}$

Abstract vector spaces

In traditional, abstract approach to linear algebra:

- Traditional: don't define vectors as sequences [1,2,3] or even functions {`a:1, b:2, c:3`}.
- Traditional: define a *vector space* over a field $\mathbb{F}$ to be any set $\mathcal{V}$ that is equipped with
  - an *addition* operation,and
  - an *additive identity* (the zero vector)
  - an *additive inverse* operation (i.e. negation),
  - a *scalar-multiplication* operation

  satisfying certain axioms (commutative, associative, and distributive laws, what happens when scalar is zero or one)

Example: All functions with domain $\{x \in \mathbb{R} \ : \ 0 \leq x \leq 1\}$ is a vector space over $\mathbb{R}$:

- For such a function f and a real number α, the function αf is defined by the rule $(\alpha f)(x) = \alpha\, f(x)$
- For two such functions f and g, $f + g$ is the function defined by the rule $(f + g)(x) = f(x) + g(x)$.
- The operations are commutative and associative.
- For a function f, $-f$ is the function defined by the rule $(-f)(x) = -(f(x))$.
- The vector $\mathbf{0}$ is the function f that maps every value to 0.

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  satisfying certain axioms (commutative, associative, and distributive laws)

Abstract approach has the advantage that it avoids committing to specific structure for vectors.

I avoid abstract approach in this class because more concrete notion of vectors is helpful in developing intuition.

What vector spaces do we study in this class?

For any field $\mathbb{F}$ and any set D, $\mathbb{F}^D$ is a vector space:

- Vector addition is a function $add : \mathbb{F}^D \times \mathbb{F}^D \longrightarrow \mathbb{F}^D$
- Scalar-vector multiplication is a function $scalar\_mul : \mathbb{F} \times \mathbb{F}^D \longrightarrow \mathbb{F}^D$

(In this class, we usually think only about finite D.)

However, this is not the only kind of vector space we consider.

Consider any subspace $\mathcal{V}$ of $\mathbb{F}^D$:

By Properties V2 and V3, the addition and scalar-multiplication operations defined for $\mathbb{F}^D$ can be viewed as addition and scalar-multiplication operations for $\mathcal{V}$:

- By Property V2, when we restrict the domain of *add* to $\mathcal{V} \times \mathcal{V}$, we can restrict the co-domain to $\mathcal{V}$.
- By Property V3, when we restrict the domain of *scalar_mul* to $\mathbb{F} \times \mathcal{V}$, we can restrict the co-domain to $\mathcal{V}$
- These operations satisfy commutative, associative, distributive laws.

By Property V1, the zero vector is included in $\mathcal{V}$.

So $\mathcal{V}$ is a vector space.

Conclusion: Any subspace of a vector space is itself a vector space.

Vector Space examples

Examples of vector spaces:

- $\mathbb{R}^3$
- $GF(2)^{\{\text{'a'},\text{'b'},\text{'c'}\}}$

Examples of subspaces of $\mathbb{R}^3$:

- $\{\mathbf{v} \ : \ [1,2,3] \cdot \mathbf{v} = 0\}$
- $\mathbb{R}^3$
- $\{\mathbf{0}\}$
- $\{\mathbf{v} \ : \ [1,2,3] \cdot \mathbf{v} = 0, [4,5,6] \cdot \mathbf{v} = 0\}$
- $\text{Span } \{[5,6,7],[8,9,10]\}$