

Morpheus: Neo, sooner or later you're going to realize, just as I did, that there's a difference between knowing the path, and walking the path.



Why CS 53?

- ▶ Making linear algebra more concrete.
- ▶ Making it more relevant.
- ▶ Taking advantage of your programming ability:
 - ▶ you can write programs that actually do semi-useful stuff
 - ▶ writing a program helps you understand
 - ▶ programming experience gives you a useful perspective

Goal: That vectors and matrices and orthogonal projection and eigenvalues and become things you interact with and use almost as easily as you use a pencil or a computer or a while-loop or recursion.

Aim: To understand some very cool but more advanced concepts/applications: *k*-Means Clustering, Principal Components Analysis, image deblurring, image compression, ...

CS53 Work categories

1. **Quizzes:** In class, on your own, including what are ordinarily considered midterm exams
2. **Class activities:** Like quizzes, but these are ungraded and you are encouraged to work with other students.
3. **Homework:** After every lecture (except the last)
4. **Labs:** Once a week for two hours
5. **Final exam:** December 16, 9:00-12:00

The workload is not high (compared to average CS class) but it is steady, and if you fall behind, you will be lost.

Quizzes

Almost every day of class there will be a quiz.

Quiz starts at the beginning of class.

Quiz ends at the end of class.

In between, there will be other things going on.

Quizzes are not intended to be challenging. They are intended to keep you on top of the material.

First quiz question

Tell me something interesting about yourself.

Homework

After almost every lecture, you will be assigned a reading in the textbook and a few related problems.

Problems due before the next class.

Two kinds of problems:

1. Problems turned in on paper (CS053 hand-in bin on second floor of CIT)
2. Problems graded and turned in digitally.

First homework

I will assign the first homework soon after this class. It will consist of questions about the course missive.

(You must do this homework but we won't actually grade it count it towards your grade.)

Labs

Every week there will be a lab. You are assigned a lab session, and you are expected to go to your lab session.

Labs start tonight. The first week only, we will have lab sessions tonight and Thursday afternoon, with a make-up session Sunday.

1. Today 7-9 pm (CIT 201)
2. Thursday 4-6 pm (CIT 201)
3. Sunday 2:00-4:00 pm (CIT 201)

Only go to one of these! If you cannot make it to any of these, let us know.

Regular weekly labs (starting next week):

1. Monday 7-9 pm
2. Tuesday, 4-6 pm
3. Tuesday, 6:30-8:30 pm

Communication

Email most questions, etc. to `cs053tas@cs.brown.edu`, which goes to all course staff, including professor.

See course web page (`csmatrix.org`) forwards to it), which has course calendar, etc. Emails (e.g. with homeworks and announcements) to you if you have registered for course or have the course in your shopping cart.

If you want to receive course communication but are not planning to register, email me: `klein@brown.edu`

To make an appointment with me, talk to me after class or email me.

Every student is required to come at least once to my office hours. I will organize group field trips. My office is CIT 111111111.

Course environment

My aim is that this course provide a welcoming and inclusive environment.

I expect everyone, including course staff and students, to be respectful and kind to each other.

I want people to feel comfortable asking questions in class. However, sometimes I will need to move on. If I cannot get to your question, please write it down and let's talk about it after class.

I want the environment to be relaxed and fun. I sometimes make jokes but I don't want anyone to feel that the joke is at their expense.

Keeping up

The workload is not huge but it is continuous and cumulative.

You need to keep up or things will not go well.

You need to allocate an hour or two between each class to

- ▶ do the homework,
- ▶ review the material from last lecture, and
- ▶ test your knowledge.

[0] The Function (and other mathematical and computational preliminaries)

Set terminology and notation

Set: an unordered collection of objects. **Example:** $\{\heartsuit, \spadesuit, \clubsuit, \diamondsuit\}$

\in : indicates that an object belongs to a set (equivalently, that the set *contains* the object). For example, $\heartsuit \in \{\heartsuit, \spadesuit, \clubsuit, \diamondsuit\}$.

$A \subseteq B$: Read this as “ A is a **subset** of B ”. This means A and B are sets, and every element of A is also an element of B .

$A = B$: Two sets are equal if they contain exactly the same elements. (There is no order among elements of a set.)

A convenient way to prove that A and B are equal is to prove that each is a subset of the other. The proof often consists of two parts:

1. a proof that $A \subseteq B$, and
2. a proof that $B \subseteq A$.

Set expressions

In Mathese, we would write “the set of nonnegative numbers” like this:

$$\{x \in \mathbb{R} : x \geq 0\}$$

Read this as “The set of consisting of all elements x of the set of real numbers such that x is greater than or equal to 0” ; The colon stands for “such that”.

There are two parts to this set expression:

- ▶ *the part before the colon*: This part specifies where the elements of the set come from, and introduces a variable or variables that can be used in the second part.
- ▶ *the part after the colon*: This gives a rule that restricts which elements specified in the first part actually get to make it into the set.

The analogous Python expression is a *set comprehension*:

```
>>> S = {-4, 4, -3, 3, -2, 2, -1, 1, 0}
```

```
>>> {x for x in S if x >= 0}
```

```
{0, 1, 2, 3, 4}
```

Set expressions

Instead of

$$\{x \in \mathbb{R} : x \geq 0\}$$

you might see just

$$\{x : x \geq 0\}$$

if it is considered clear what kind of values x is supposed to take on.

Another example:

$$\{x : x^2 - \frac{5}{6}x + \frac{1}{6} = 0\}$$

This time, the set consists of just two numbers, $\frac{1}{2}$ and $\frac{1}{3}$.

Set terminology and notation

Cardinality: If a set S is not infinite, we use $|S|$ to denote the number of elements or *cardinality* of the set.

For example, the set $\{\heartsuit, \spadesuit, \clubsuit, \diamondsuit\}$ has cardinality 4.

Set terminology and notation: Cartesian product

$A \times B$ is the set of all pairs (a, b) where $a \in A$ and $b \in B$.

Example: for $A = \{1, 2\}$ and $B = \{\heartsuit, \spadesuit, \clubsuit, \diamondsuit\}$, $A \times B$ is

$$\{(1, \heartsuit), (2, \heartsuit), (1, \spadesuit), (2, \spadesuit), (1, \clubsuit), (2, \clubsuit), (1, \diamondsuit), (2, \diamondsuit)\}$$

Named for René Descartes. We will meet him later.

Set terminology and notation: Cartesian product

Question: What is the cardinality of $A \times B$ where $A = \{1, 2\}$ and $B = \{\heartsuit, \spadesuit, \clubsuit, \diamondsuit\}$?

Answer: 8

Set terminology and notation: Cartesian product

If A and B are finite sets then $|A \times B| = |A| \times |B|$.

Question: What is the cardinality of $\{1, 2, 3, \dots, 10, J, Q, K\} \times \{\heartsuit, \spadesuit, \clubsuit, \diamondsuit\}$?

Answer: 52

Tuples in set expressions

The set expression

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2\}$$

denotes the set of all pairs of real numbers in which the second element of the pair is the square of the first.

This set expression might be abbreviated as

$$\{(x, y) : y = x^2\}$$

where you are supposed to guess from context that x and y range over real numbers.

Another example:

$$\{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : x \geq 0, y \geq 0, z \geq 0\}$$

This is the set of triples consisting of nonnegative real numbers.

To include (x, y, z) , *all* the conditions to right of colon must be satisfied. (You can read the comma between the conditions as “and”.)

We might abbreviate that set expression as $\{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}$

What is a function?

Informally, for each *input* element in a set A , a *function* assigns a single *output* element from another set B .

- ▶ A is called the *domain* of the function
- ▶ B is called the *co-domain*

Formally, a function from A to B is a subset of the Cartesian product $A \times B$ such that, for each element $a \in A$, there is exactly one element $b \in B$ such that the subset contains (a, b) .

Example: The function with domain $\{1, 2, 3, \dots\}$ that doubles its input is the set

$$\{(1, 2), (2, 4), (3, 6), (4, 8), \dots\}$$

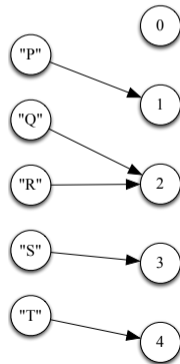
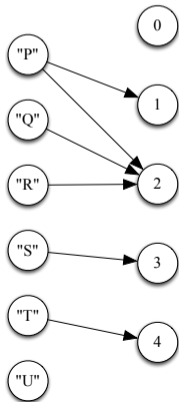
Example: The function with domain $\{1, 2, 3, \dots\} \times \{1, 2, 3, \dots\}$ that multiplies the numbers forming its input is

$$\{((1, 1), 1), (1, 2), 2), ((1, 3), 3), \dots, ((2, 1), 2), ((2, 2), 4), ((2, 3), 6), \dots\}$$

Diagrams of functions

Diagram for subset of a Cartesian product or a function:

- ▶ The subset $\{("P", 1), ("P", 2), ("Q", 2), ("R", 2), ("S", 3), ("T", 4)\}$ of $\{"P", "Q", "R", "S", "T", "U"\} \times \{0, 1, 2, 3, 4\}$
- ▶ The function $\{("P", 1), ("Q", 2), ("R", 2), ("S", 3), ("T", 4)\}$ with domain $\{"P", "Q", "R", "S", "T"\}$ and co-domain $\{0, 1, 2, 3, 4\}$



Activity

Draw two diagrams of subsets of the Cartesian product $\{ "P", "Q" \} \times \{0, 1\}$ that illustrate the two ways a subset of the Cartesian product can *fail* to be a function.

Some function terminology

Definition: The output of a given input is called the *image* of that input. The image of q under a function f is denoted $f(q)$.

If $f(q) = r$, we say q maps to r under f . In Mathese, we write this as $q \mapsto r$.

The set from which all the outputs are chosen is called the *co-domain*.

We write

$$f : A \longrightarrow B$$

when we want to say that f is a function with *domain* A and *co-domain* B .

Note: When we define a function, we have some flexibility in the choice of co-domain. There might be elements of the co-domain that are *not* images of any elements of the domain.

Example: Caesar's Cryptosystem

Each letter is mapped to one three places ahead, wrapping around, so MATRIX would map to PDWULA.

The function mapping letter to letter can be written as:

$$\{('A', 'D'), ('B', 'E'), ('C', 'F'), ('D', 'G'), \dots, ('W', 'Z'), ('X', 'A'), ('Y', 'B'), ('Z', 'C')\}$$

Both the domain and co-domain are $\{A, B, \dots, Z\}$.

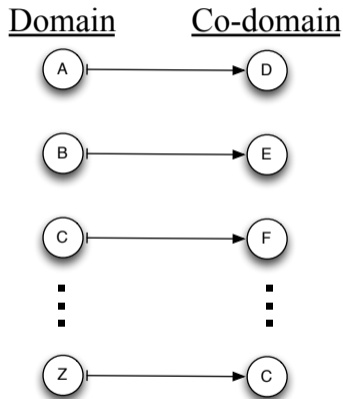


Image of a function

Definition: The *image* of a function is the set of all images of inputs. Mathese: $\text{Im } f$

Example: Cosine Function $\cos(x)$

$\cos : \mathbb{R} \rightarrow \mathbb{R}$, which means the domain is \mathbb{R} and the co-domain is \mathbb{R}

The image of $\cos(x)$, $\text{Im } \cos$, is $\{x \in \mathbb{R} : -1 \leq x \leq 1\}$,
which is not the same as the co-domain

Example: The image of the Caesar encryption function is $\{A, B, C, \dots, Z\}$,
which is the same as the co-domain.

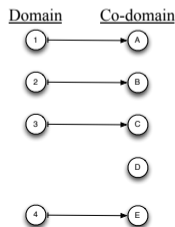
Example: a function $f : \{1, 2, 3, 4\} \rightarrow \{ 'A', 'B', 'C', 'D', 'E' \}$

The image of f is $\text{Im } f = \{ 'A', 'B', 'C', 'E' \}$

'D' is in the co-domain but not in the image.

- ▶ Some people use “range” to mean co-domain.
- ▶ Some people use “range” to mean image.

Because it is used in both ways, I prefer to avoid the word.



Set of functions with given domain and co-domain

Definition: For sets F and D , F^D denotes all functions from D to F .

Example: The set of functions from the set W of words to the set \mathbb{R} of real numbers is \mathbb{R}^W .

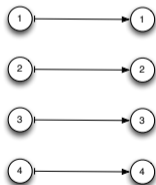
Proposition: For finite sets F and D , $|F^D| = |F|^{|D|}$.

Identity function

Identity function: for any domain D ,

$$id_D : D \longrightarrow D$$

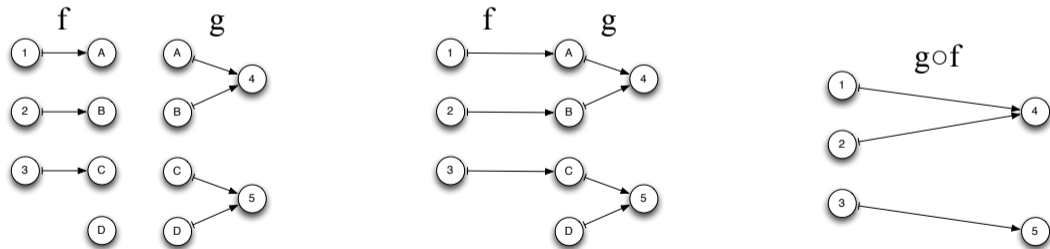
maps each domain element d to itself.



Composition

Definition: For functions $f : A \rightarrow B$ and $g : B \rightarrow C$, the *functional composition* of g and f is the function $(g \circ f) : A \rightarrow C$ defined by $(g \circ f)(x) = g(f(x))$

Example: $f : \{1, 2, 3\} \rightarrow \{A, B, C, D\}$ and $g : \{A, B, C, D\} \rightarrow \{4, 5\}$



Example: Composition of $g(y) = y^2$ and $f(x) = x + 1$ is $(g \circ f)(x) = (x + 1)^2$.

Function composition

Example: Define the function

$$f : \{A, B, C, \dots, Z\} \longrightarrow \{0, 1, 2, \dots, 25\}$$

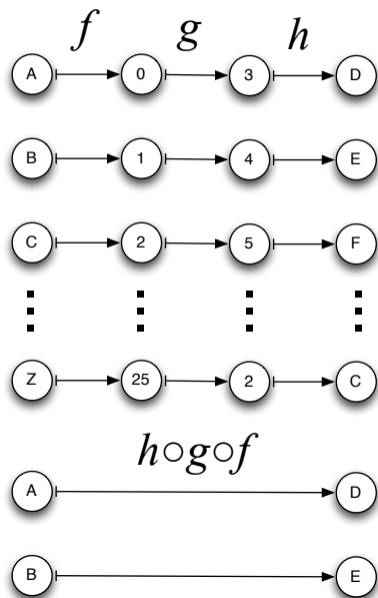
by $A \mapsto 0, B \mapsto 1, C \mapsto 2, \dots, Z \mapsto 25$

Define g on the domain/co-domain

$$\{0, 1, 2, \dots, 25\} \text{ by } g(x) = (x + 3) \bmod 26$$

Define h with domain $\{0, 1, 2, \dots, 25\}$ and co-domain $\{A, \dots, Z\}$ such that $0 \mapsto A, 1 \mapsto B, \text{ etc.}$

Then $h \circ (g \circ f)$ is the Caesar cypher.



Associativity of function composition

Proposition: $h \circ (g \circ f) = (h \circ g) \circ f$

Proof: for any element x of domain of f :

$$\begin{aligned}(h \circ (g \circ f))(x) &= h((g \circ f)(x)) \text{ by definition of } h \circ (g \circ f) \\ &= h(g(f(x))) \text{ by definition of } g \circ f \\ &= (h \circ g)(f(x)) \text{ by definition of } h \circ g \\ &= ((h \circ g) \circ f)(x) \text{ by definition of } (h \circ g) \circ f\end{aligned}$$

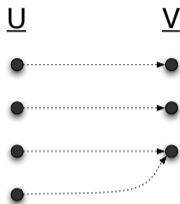
Functional inverse

Definition: Functions f and g are *functional inverses* if $f \circ g$ and $g \circ f$ are defined and are identity functions.

A function that has an inverse is *invertible*.

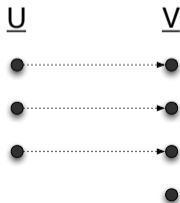
One-to-one

Definition: $f : D \rightarrow F$ is *one-to-one* if $f(x) = f(y)$ implies $x = y$.



NOT ONE-TO-ONE

Definition: $f : D \rightarrow F$ is *onto* if for every $z \in F$ there exists an a such that $f(a) = z$.



NOT ONTO

One-to-one

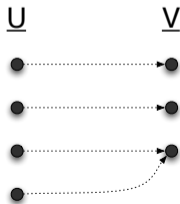
Definition: $f : D \rightarrow F$ is *one-to-one* if $f(x) = f(y)$ implies $x = y$.

Proposition: Invertible functions are one-to-one.

Proof: Assume an invertible function f is not one-to-one. So there exists $x_1 \neq x_2$ where $f(x_1) = f(x_2) = y$.

Then $f^{-1}(y) = x_1$ but $f^{-1}(y) = x_2$, and both cannot be true, by the definition of function.

QED



NOT ONE-TO-ONE

Onto

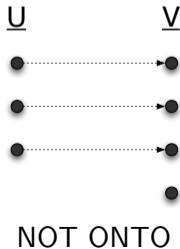
Definition: $f : D \rightarrow F$ is *onto* if for every $z \in F$ there exists an element $a \in D$ such that $f(a) = z$.

Proposition: Invertible functions are onto

Proof Assume an invertible function f is not onto. So there exists element \hat{y} in co-domain such that for no x does $f(x) = \hat{y}$.

But $f^{-1}(\hat{y}) = \hat{x}$ for some \hat{x} , and by the definition of the inverse, $f(\hat{x}) = \hat{y}$, a contradiction.

QED



One-to-one and onto = invertible

Function Invertibility Theorem:

A function f is invertible if and only if it is one-to-one and onto.

Previous two propositions show that every invertible function is one-to-one and onto. It is not hard to prove that a function that is one-to-one and onto is invertible.

Activity

Draw two diagrams of subsets of the Cartesian product $\{ "P", "Q" \} \times \{0, 1\}$ that illustrate the two ways a function can *fail* to be invertible.