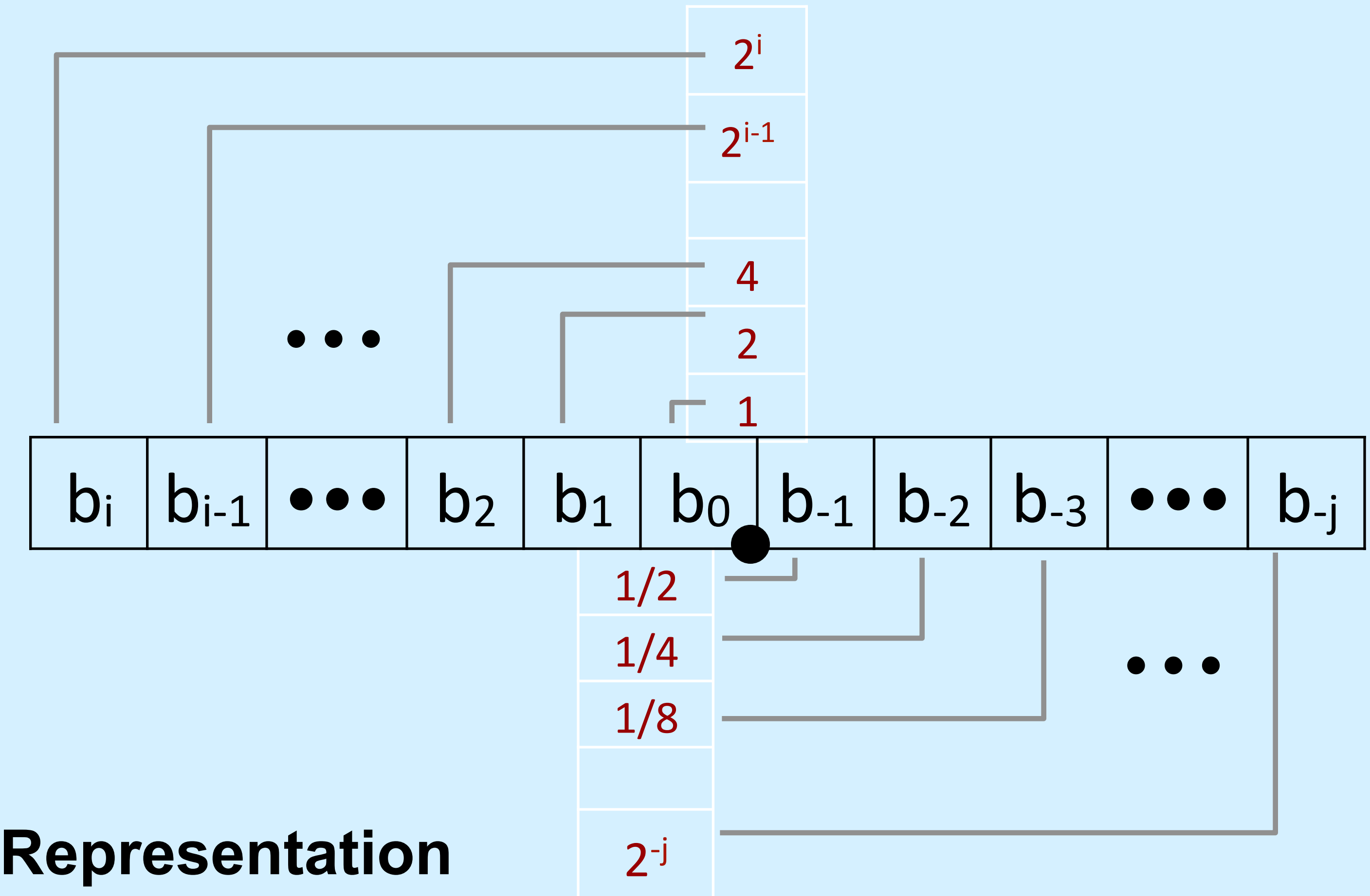
CS 33

Data Representation (Part 3)

Fractional binary numbers

- What is 1011.101_2?

Fractional Binary Numbers



- **Representation**
  - **bits to right of "binary point" represent fractional powers of 2**
  - **represents rational number:** $\sum_{k=-j}^{i} b_k \times 2^k$

Representable Numbers

- **Limitation #1**
  - **can exactly represent only numbers of the form n/2^k**
    - **other rational numbers have repeating bit representations**
  - **value representation**
    - **1/3** $0.0101010101[01]\ldots_2$
    - **1/5** $0.001100110011[0011]\ldots_2$
    - **1/10** $0.0001100110011[0011]\ldots_2$

- **Limitation #2**
  - **just one setting of decimal point within the *w* bits**
    - **limited range of numbers (very small values? very large?)**

IEEE Floating Point

- IEEE Standard 754
  - established in 1985 as uniform standard for floating point arithmetic
    - » before that, many idiosyncratic formats
  - supported by all major CPUs

- Driven by numerical concerns
  - nice standards for rounding, overflow, underflow
  - hard to make fast in hardware
    - » numerical analysts predominated over hardware designers in defining standard

Floating-Point Representation

- **Numerical Form:**

$$(-1)^s \; M \;\; 2^E$$

  - **sign bit s determines whether number is negative or positive**
  - **significand M normally a fractional value in range [1.0,2.0)**
  - **exponent E weights value by power of two**
- **Encoding**
  - **MSB `s` is sign bit s**
  - **`exp` field encodes E (but is not equal to E)**
  - **`frac` field encodes M (but is not equal to M)**

`s`	`exp`	`frac`

Precision options

- **Single precision: 32 bits**

s	exp	frac
1	8-bits	23-bits

- **Double precision: 64 bits**

s	exp	frac
1	11-bits	52-bits

- **Extended precision: 80 bits (Intel only)**

s	exp	frac
1	15-bits	64-bits

“Normalized” Values

- When: exp ≠ 000…0 and exp ≠ 111…1
- Exponent coded as biased value: E = Exp – Bias
  - exp: unsigned value exp
  - bias = 2^{k-1} - 1, where k is number of exponent bits
    - » single precision: 127 (Exp: 1…254, E: -126…127)
    - » double precision: 1023 (Exp: 1…2046, E: -1022…1023)
- Significand coded with implied leading 1: M = $1.xxx…x_2$
  - xxx…x: bits of frac
  - minimum when frac=000…0 (M = 1.0)
  - maximum when frac=111…1 (M = 2.0 – ε)
  - get extra leading bit for “free”

Normalized Encoding Example

- **Value:** `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**

  M = $1.\underline{1101101101101}_2$

  `frac` = $\underline{1101101101101}0000000000_2$

- **Exponent**

  E = **13**

  $bias$ = **127**

  exp = **140** = 10001100_2

- **Result:**

s	exp	frac
0	10001100	11011011011010000000000

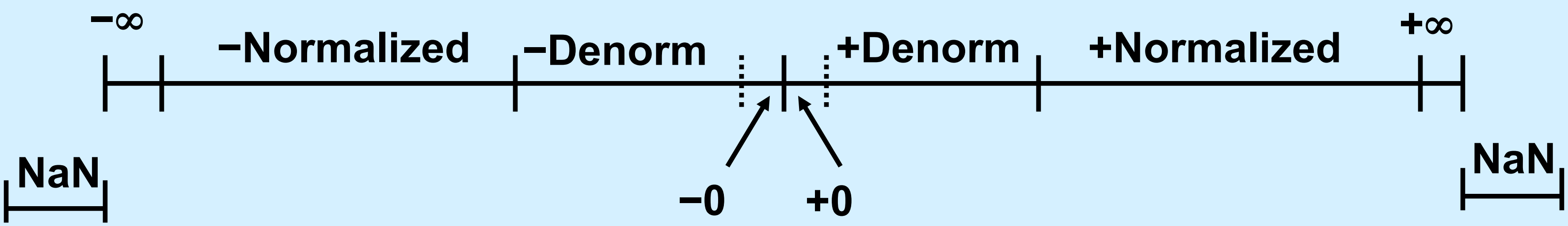
Denormalized Values

- Condition: `exp = 000…0`
- Exponent value: E = –Bias + 1 (instead of E = 0 – Bias)
- Significand coded with implied leading 0:
  M = $0.xxx\ldots x_2$
  - `xxx…x`: bits of `frac`
- Cases
  - `exp` = 000…0, `frac` = 000…0
    » represents zero value
    » note distinct values: +0 and –0 (why?)
  - `exp` = 000…0, `frac` ≠ 000…0
    » numbers closest to 0.0
    » equispaced

Special Values

- Condition: `exp` = 111…1
- Case: `exp` = 111…1, `frac` = 000…0
  - represents value ∞ (infinity)
  - operation that overflows
  - both positive and negative
  - e.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞
- Case: `exp` = 111…1, `frac` ≠ 000…0
  - not-a-number (NaN)
  - represents case when no numeric value can be determined
  - e.g., sqrt(–1), ∞ − ∞, ∞ × 0

Visualization: Floating-Point Encodings



Tiny Floating-Point Example

s	exp	frac
1	4-bits	3-bits

- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the `frac`

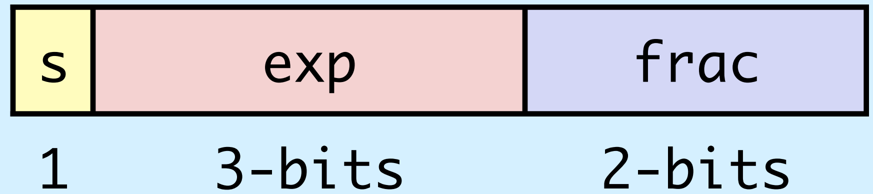
- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

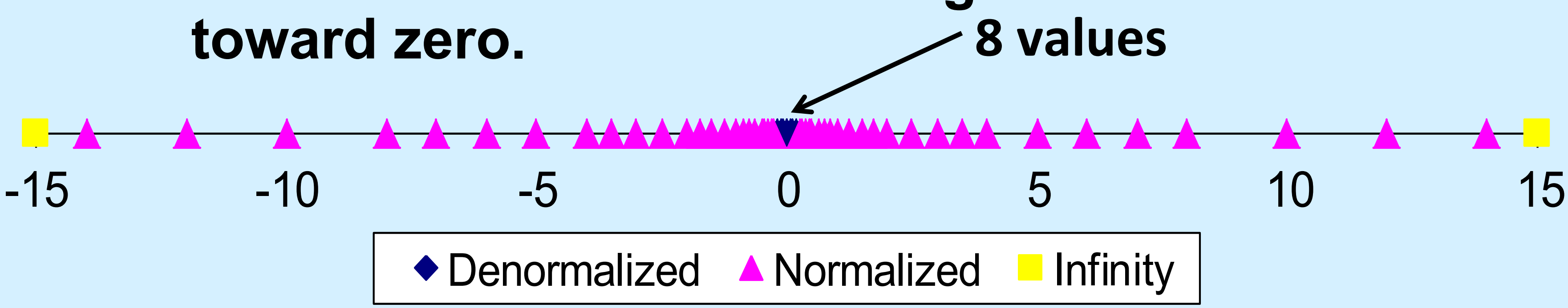
	s exp frac	E	Value	
	0 0000 000	-6	0	
	0 0000 001	-6	1/8*1/64 = 1/512	closest to zero
Denormalized numbers	0 0000 010	-6	2/8*1/64 = 2/512	
	…			
	0 0000 110	-6	6/8*1/64 = 6/512	
	0 0000 111	-6	7/8*1/64 = 7/512	largest denorm
	0 0001 000	-6	8/8*1/64 = 8/512	smallest norm
	0 0001 001	-6	9/8*1/64 = 9/512	
	…			
	0 0110 110	-1	14/8*1/2 = 14/16	
	0 0110 111	-1	15/8*1/2 = 15/16	closest to 1 below
Normalized numbers	0 0111 000	0	8/8*1 = 1	
	0 0111 001	0	9/8*1 = 9/8	closest to 1 above
	0 0111 010	0	10/8*1 = 10/8	
	…			
	0 1110 110	7	14/8*128 = 224	
	0 1110 111	7	15/8*128 = 240	largest norm
	0 1111 000	n/a	inf	

Distribution of Values

- **6-bit IEEE-like format**
  - **e = 3 exponent bits**
  - **f = 2 fraction bits**
  - **bias is 2^{3-1}-1 = 3**

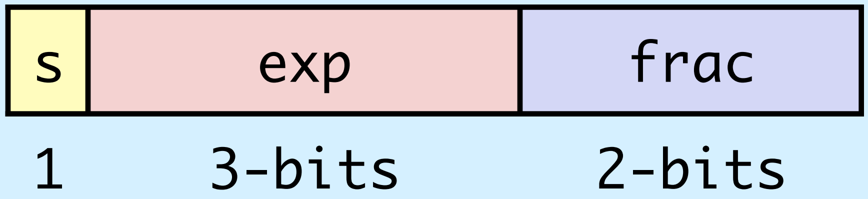


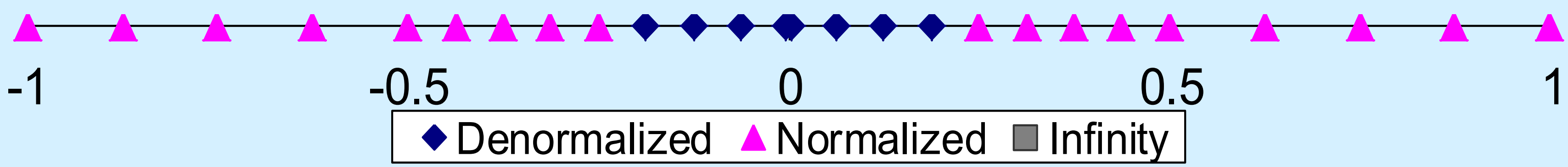
- **Notice how the distribution gets denser toward zero.**



Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - bias is 3





Quiz 1

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - bias is 3

s	exp	frac
1	3-bits	2-bits

What number is represented by 0 011 10?

a) 12

b) 1.5

c) .5

d) none of the above

Floating-Point Operations: Basic Idea

- $\mathtt{x} +_{\mathtt{f}} \mathtt{y} = \mathtt{Round(x + y)}$

- $\mathtt{x} \times_{\mathtt{f}} \mathtt{y} = \mathtt{Round(x} \times \mathtt{y)}$

- **Basic idea**
  - first **compute exact result**
  - make it fit into desired precision
    - » possibly overflow if exponent too large
    - » possibly **round to fit into** `frac`

Rounding

- Rounding modes (illustrated with $ rounding)

	$1.40	$1.60	$1.50	$2.50	–$1.50
towards zero	$1	$1	$1	$2	–$1
round down (−∞)	$1	$1	$1	$2	–$2
round up (+∞)	$2	$2	$2	$3	–$1
nearest even (default)	$1	$2	$2	$2	–$2

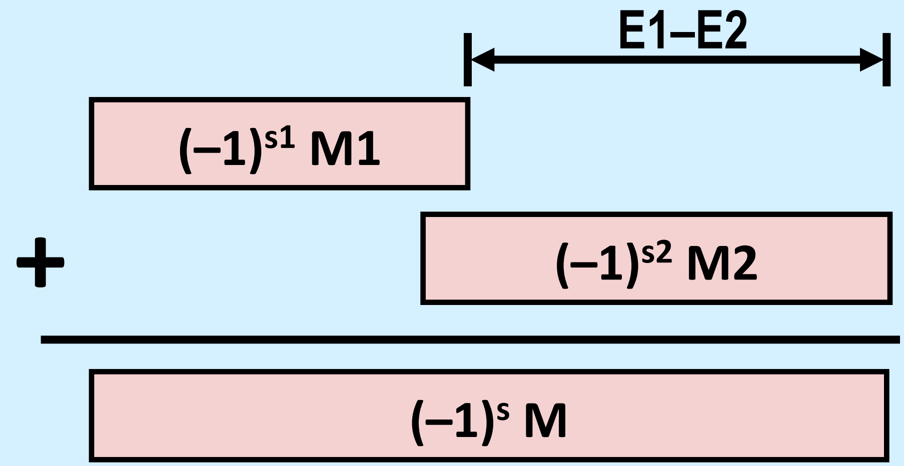
Floating-Point Multiplication

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ \ x\ \ \ (-1)^{s2}\ M2\ \ 2^{E2}$
- Exact result: $(-1)^{s}\ M\ \ 2^{E}$
  - sign s: s1 ^ s2
  - significand M: M1 x M2
  - exponent E: E1 + E2

- Fixing
  - if M ≥ 2, shift M right, increment E
  - if E out of range, overflow (or underflow)
  - round M to fit `frac` precision
- Implementation
  - biggest chore is multiplying significands

Floating-Point Addition

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
  - –assume $E1 > E2$



- **Exact result:** $(-1)^{s}\ M\ \ 2^{E}$
  - –sign s, significand M:
    - » result of signed align & add
  - –exponent E: E1
- **Fixing**
  - –if M ≥ 2, shift M right, increment E
  - –if M < 1, shift M left k positions, decrement E by k
  - –overflow if E out of range
  - –round M to fit `frac` precision

Floating Point in C

- C guarantees two levels
  - `float` single precision
  - `double` double precision
- Conversions/casting
  - casting between `int`, `float`, and `double` changes bit representation
  - `double`/`float` → `int`
    - » truncates fractional part
    - » like rounding toward zero
    - » not defined when out of range or NaN: generally sets to TMin
  - `int` → `double`
    - » exact conversion, as long as `int` has ≤ 53-bit word size
  - `int` → `float`
    - » will round according to rounding mode

Quiz 2

Suppose *f*, declared to be a `float`, is assigned the largest possible floating-point positive value (other than +∞). What is the value of *g = f+1.0*?

a) f

b) +∞

c) NAN

d) 0

Float is not Rational …

- Floating addition
  - commutative: $a +^f b = b +^f a$
    - » yes!
  - associative: $a +^f (b +^f c) = (a +^f b) +^f c$
    - » no!
      - $2 +^f (1e10 +^f -1e10) = 2$
      - $(2 +^f 1e10) +^f -1e10 = 0$

Float is not Rational …

- **Multiplication**
  - commutative: $a *^f b = b *^f a$
    - » yes!
  - associative: $a *^f (b *^f c) = (a *^f b) *^f c$
    - » no!
      - $1e20 *^f (1e20 *^f 1e\text{-}20) = 1e20$
      - $(1e20 *^f 1e20) *^f 1e\text{-}20 = +\infty$

Float is not Rational …

- More …
  - multiplication distributes over addition:
    $a *^f (b +^f c) = (a *^f b) +^f (a *^f c)$
    - » no!
    - » $1e20 *^f (1e20 +^f -1e20) = 0$
    - » $(1e20 *^f 1e20) +^f (1e20 *^f -1e20) = NaN$
  - loss of significance:

    x=y+1

    z=2/(x-y)

    z==2?
    - » not necessarily!
      - consider y = 1e20

