Bounds

**Note:** For the following definitions, we assume a probability space \((S, p)\).

**Markov’s Inequality:** Given a non-negative random variable \(X\) and a constant \(a > 0\):

\[
p(X \geq a) \leq \frac{E[X]}{a}.
\]

(Proven.)

**Chebyshev’s Inequality:** Given a random variable \(X\) and a constant \(a > 0\):

\[
p(|X - E[X]| \geq a) \leq \frac{V[X]}{a^2}.
\]

(Not proven.)

**Weak Law of Large Numbers (WLLN):** Suppose we have \(n\) independent identical random variables \(X_1, X_2, \ldots, X_n\). Then for any constant \(a > 0\):

\[
\lim_{n \to \infty} p\left(\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - E[X_i]\right| \geq a\right) = 0.
\]

(Not proven.)

**Proposition:** Given a random variable \(X\),

\[
V[aX] = a^2V[X].
\]

(Not proven. Useful for the proof of Chebyshev’s Bound and WLLN.)

Graph Theory

**Definition:** An undirected simple graph \(G = (V, E)\) is a pair, where \(V\) is a set and \(E\) is a set of unordered pairs of elements of \(V\). We call \(V\) the set of vertices and \(E\) the set of edges.

**Note:** For the following definitions and propositions, we will deal with only undirected simple graphs.

**Proposition:** There are \(2^{\binom{n}{2}}\) distinct graphs on \(n\) vertices. (Proven.)

**Proposition:** The number of distinct graphs on \(n\) vertices up to relabeling (i.e. up to isomorphism) is asymptotically \(\frac{2^{\binom{n}{2}}}{n!}\). (Not proven.)

**Definition:** If \(\{u, v\} \in E\), we call \(u\) and \(v\) neighbors. For a given vertex \(v\), we call the number of neighbors of \(v\) the degree of \(v\), denoted \(\text{deg}(v)\).

**Proposition:** For any graph on \(n \geq 2\) vertices, two vertices have the same degree. (You’ve proven this already—it is the dance partner problem.)
Proposition: For any graph $G = (V, E)$:

$$\sum_{v \in V} \deg(v) = 2|E|.$$  

(Not proven.)