Number Systems

- $\mathbb{Z}$ Integers
- $\mathbb{Q}$ Rationals
- $\mathbb{R}$ Reals

*Careful:* When making a proposition, must specify in which space we are working.

**Prop:** For all integers $a$, there exists an integer $b$ such that $a + b = 0$.

**Prop:** For all $a$, there exists $b$ such that $ab = 1$. This is not true over the integers (e.g. $a = 3$, $b$ must be $\frac{1}{3} \notin \mathbb{Z}$). This is also not true over $\mathbb{Q}$, because it does not apply to all elements of $\mathbb{Q}$. It does work for “Over $\mathbb{Q}$, for all $a \neq 0$,” then this is true.

Sets

A set is a collection of distinct, unordered objects (elements). Sets are denoted by curly brackets, with its elements separated by commas.

- Example: $S = \{1, 2, t, \text{purple}, 6, 5\} = \{5, \text{purple}, 1, t, 2, 6\}$.
- Bad Example: $S = \{2, 2, 3\}$ (bad notation, but we interpret this to be the set containing the two elements 2 and 3.

$t \in S$, means “$t$ is an element of $S$.” $k \notin S$ means “$k$ is not an element of $S$.”

**Set Builder Notation:** Define a set via a rule.

- $\mathbb{Z} = \{x \mid 1 \leq x \leq 10, x \in \mathbb{Z}\} = \{1, 4, 5, 9, 10, 6, 7, 2, 3, 8\}$.

**Cardinality:** Define the cardinality of a set $X$ to be the number of elements in $X$ if $X$ is finite, otherwise we say the cardinality is infinite. We write $|X|$ as the cardinality of $X$.

**Union:** Define the union of $A$ and $B$ as $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

An example is $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, and $A \cup B = \{1, 2, 3, 4, 5\}$. Note 3 is not repeated.
0/1 Strings: We can write sets as 0/1 strings, i.e. $A$ from before can be written as 11100 and $B$ is 00111 and $A \cup B = 11111$.

Subset: Denoted $A \subseteq B$, $A$ is a subset of $B$ if every element of $A$ is also an element of $B$.

- Example: Let $T = \{2, 3\}$. $T \subseteq \mathbb{Z}$. $\mathbb{Z} \not\subseteq S$. Every set is a subset of itself: $T \subseteq T$. However, a set is not a proper subset of itself.

Empty Set: The empty set is the set with no objects, denoted {} or $\emptyset$. The empty set is a subset of all sets.

Intersection: Define the intersection of $A$ and $B$ as $A \cap B = \{x | x \in A \text{ and } x \in B\}$.

An example is $A = \{1, 2, 3\}$ and $B = \{4\}$, and $A \cap B = \emptyset$. Note since there are no overlaps, the intersection is the empty set.

Power Set: Given a set $X$, define the power set of $X$ as the set of all subsets of $X$, denoted $\mathcal{P}(X)$ or $2^X$.

The number of binary strings of length $n$ is equal to $2^n$.

$D = \{\text{purple, 4, } \{3, 4\}, 6\}$ has size four ($\{3, 4\}$ is its own object, which is independent of the object 4 alone).

Claim: If $X$ is a set with $n$ objects, then $X$ has $2^n$ subsets.

- Example: There are 8 0/1 strings of length 3. $X = \{a, b, c\}$. Subsets of $X$:
  1. $\{a, b, c\}$ → 111
  2. $\{a, b\}$ → 110
  3. $\{a, c\}$ → 101
  4. $\{b, c\}$ → 011
  5. $\{a\}$ → 100
  6. $\{b\}$ → 010
  7. $\{c\}$ → 001
  8. $\{}$ → 000

Impose an “ordering” on the elements in the set, and use the ordered strings to denote whether or not each element in the set is included in the subset (1 means the element is in, 0 means the element is out).