Induction

\[ P(n) \ \forall n \geq b, n \in \mathbb{Z} \]

- Base Case: \( p(b) \)
- Inductive step: \( p(k) \implies p(k + 1) \)

**Ex 1** \( p(n) : 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \)

**Proof.** Base Case: \( p(1) \)
\[ 1 = \frac{1(1+1)}{2} \]

Inductive step: Assume \( p(k) \) is true for a fixed but arbitrary value \( k \)

Claim: Want to show \( p(k + 1) \) is true

**Proof.**
\[ p(k + 1) : 1 + 2 + 3 + \ldots + k + k + 1 = \frac{k(k + 1)}{2} + \frac{2 \cdot (k + 1)}{2} \]
by Inductive Hypothesis
\[ = \frac{(k + 1)(k + 2)}{2} \]

**Ex 2:** \( p(n) \): number of subsets of an \( n \) element set = \( 2^n \) for \( n > 0 \)

- \( P(0) : 1 = 2^0 \) = number of subsets of \( \emptyset \)
- \( P(1) : 2 = 2^1 \) = number of subsets of a 1 element set

Inductive step: Assume \( P(k) \) for a fixed but arbitrary value of \( k \)

Claim: \( P(k + 1) \) is true. Let \( X \) be a set with \( k + 1 \) elements. Let \( y \in X \)

How many subsets of \( X \) contain \( y \)?
\( y \) plus any subset of \( X \setminus \{y\} \)
By our I.H., there are \( 2^k \) subsets of \( X \setminus \{y\} \) so there are still \( 2^n \) when we add \( y \)

How many subsets of \( X \) do no contain \( y \)?
Any subset of \( X \setminus \{y\} \)
By our I.H., there are \( 2^k \) subsets of \( X \setminus y \)
Total number of subsets = $2^k + 2^k = 2^{k+1}$

As an example, consider a 2 element set $\{1, 2\}$ and its power set, $\{\{1, 2\}, \{1\}, \{2\}, \emptyset\}$. To go from $p(2)$ to $p(3)$, we consider the added element 3 and separate the possible subsets into two sets: ones that include 3 and ones that don’t.

The first set: $\{\{1, 2, 3\}, \{1, 3\}, \{2, 3\}, \{3\}\}$

The second set: $\{\{1, 2\}, \{1\}, \{2\}, \emptyset\}$

Note that each set is the same size! Together, they account for all of the possible subsets of $\{1, 2, 3\}$, of which there are $2^2 + 2^2 = 2^3$.

**Ex 3:** $A \cap (B_1 \cup B_2 \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cdots \cup (A \cap B_n)$

Here, the induction step is easier, but the base case is where you have to do the set element method!