More Induction

Example: Recall the claim: $A \cap (B_1 \cup B_2 \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cdots \cup (A \cap B_n)$

Base case would be the proof that $A \cap (B_1 \cup B_2) = (A \cap B_1) \cup (A \cap B_2)$. This requires set element method potentially (not just the “check-mark” we’ve been doing for base cases).

Induction step is much easier though! We want to prove this for $A \cap (B_2 \cup \cdots \cup B_{k+1})$, and we can rewrite the first $K$ sets as $C$ to get

$A \cap (C \cup B_{k+1}) = (A \cap C) \cup (A \cap B_{k+1})$ (by our base case)

And then by our IH, we can write this is $(A \cap B_1) \cup (A \cap B_2) \cdots (A \cap B_k) \cup (A \cap B_{k+1})$ thus proving our IS!

Strong Induction

$P(n) \forall n \geq b$ (base).

$P(b)$, (perhaps) $P(b + 1), P(b + 2), P(b + 3), \ldots, P(b + j)$

Assume $P(b), P(b + 1), \ldots, P(b + m)$, show $P(b + m + 1)$.

Fundamental Theorem of Arithmetic

Every $x \in \mathbb{Z}, x > 1$ can be written as a product of prime numbers.

Proof. Proof by Strong Induction

Base Case: $x = 2, 3, 4 = 2^2, 5, 6 = 2 \cdot 3, 7, 8 = 2^3$

Assume $p(1), p(2), p(3), \ldots, p(k)$ for some $k$. Prove $p(k + 1)$

Consider $k + 1$:

Case 1: $k + 1$ is prime $\checkmark$

Case 2: $k + 1$ is not prime $\implies$ $(k + 1) = a \cdot b$ with $a < (k + 1)$ and $b < (k + 1)$.

I.H. $\implies$ $a$ and $b$ can be written as a product of primes $\square$
Bad Induction

Claim: In any set of \( n \) sandwiches, they all have the same color bread.

Proof. Base case: \( n = 1 \) ✓
Inductive step: Suppose true for \( k \).
Suppose \( S \) is a set of \( k + 1 \) sandwiches. The first \( k \) have the same color by IH and so do the last \( k \). So, all of them have the same color!

Huh? How could this be the case?
It turns out for \( n = 2 \) we don’t have two overlapping sets so the two groups don’t actually have the same color! So, our induction breaks down here (the dominoes can’t all fall).

Tiling

We have an 8x8 chess board.

- Can we cover with domino tiles? Yes!
- Remove a corner. No, you can no longer tile the board because each domino must cover 2 squares, and there are now an odd number of squares.
- Remove 2 opposite corners. Now can you tile the board? No, even though the parity is the same, every domino must tile exactly one white and one black square, and we have removed two opposite corners, which are of the same color.

Now, we want to tile a \( 2^nx2^n \) board with L tiles (a 2x2 square with one corner removed).
What about tiling a board with one corner removed? Yes, this does work in general. Let \( P(n) \) be the proposition that \( 2^nx2^n \) corner can be tiled.

Proof. Proof is by induction on \( n \), for \( n \in \mathbb{Z}_{\geq 1} \).
Base case: \( P(1) \), the L tile itself tiles the 2x2 board.
Inductive hypothesis: Assume \( P(k) \) holds (and try to prove for \( P(k + 1) \)).
Consider a \( 2^{k+1}x2^{k+1} \) board with a corner removed. This board comprises four \( 2^kx2^k \) boards, one with one corner removed. We can invoke the IH on this sub-board. For
the other 3, adjacent to each other, we can remove one corner tile from each sub-
board such that the removed corners are adjacent to each other, forming the shape of an L tile.