1 Pigeonhole Principle

The pigeonhole principle states that when \( k + 1 \) objects are placed into \( k \) boxes, some box must have at least 2 objects. More formally, for any mapping \( f : A \to B \) such that \( |A| = |B| + 1 \), \( f \) is not one-to-one.

1.1 Examples

1.1.1 Socks!

Let’s say I have a drawer filled with purple socks and yellow socks. If I am taking socks blindly, how many socks do I need to remove from the drawer to ensure that I will have a pair of socks of matching color?

The pigeonhole principle tells us that if we divide our socks into a purple “box” and a yellow “box”, if we take three socks, then one of these boxes must contain at least 2 objects, representing a pair of matching socks. Thus, we must remove 3 socks to ensure that we get a pair.

1.1.2 Summing to 7

Consider the set \( X = \{1, 2, 3, 4, 5, 6\} \).

Claim: If I pick any four elements \( x_1, x_2, x_3, x_4 \) from \( X \) then some \( x_i, x_j \) such that \( 1 \leq i, j \leq 4 \), \( i \neq j \) will sum to 7.

Proof. Let’s start by dividing \( X \) into boxes (or subsets) of pairs that sum to 7:

\[
\{1, 6\} \quad \{2, 5\} \quad \{3, 4\}
\]

We are going to choose any 4 elements from \( X \): \( x_1, x_2, x_3, x_4 \). Now let’s create a mapping that maps the four elements \( x_1, x_2, x_3, x_4 \in X \) to the subset from above that contains \( x_i \). \( f \) is thus mapping 4 items to 3 subsets, so by the pigeonhole principle, \( f \) is not one-to-one and there must exist two elements in our domain \( x_i, x_j \) such that \( f(x_i) = f(x_j) \). This means that \( x_i \) and \( x_j \) belong to the same subset, so \( x_i + x_j = 7 \). \( \square \)

1.1.3 Frenemies

Claim: Suppose we have 6 people, and every pair of people in this group is either friends or enemies. Then there exists 3 people who are mutual friends or mutual enemies.

Proof. Consider person 1, denoted \( P_1 \), and their relations to the five other people. Define \( F_1 \) to be the set of people who are friends with \( P_1 \) and \( E_1 \) to be the set of people who are enemies with \( P_1 \).

For example, we could have something that looks like this:
Next we invoke the pigeonhole principle, which tells us that either $|F_1| \geq 3$ or $|E_1| \geq 3$.

WLOG, suppose $|F_1| \geq 3$. Then if any pair of people in $F_1$ are friends, $P_1$ plus that pair makes a set of 3 mutual friends, and our claim is satisfied. Otherwise, everyone in $F_1$ is/are enemies. In that case, $F_1$ represents a group of 3 mutual enemies, and again, our claim is satisfied.

1.1.4

Claim: If we have $n^2 + 1$ distinct integers then there exists an increasing or decreasing subsequence of length $n + 1$.

For example, let $n = 3$. Consider the following set of 10 distinct integers:

$\begin{align*}
1655 & \quad 1 \quad 22 \quad 15 \quad 70 \quad 7 \quad 9 \quad 3 \quad 16 \quad 1650
\end{align*}$

Then 1, 15, 70, 1650 defines a subsequence of length 4 that is continually increasing.

Proof. Consider the sequence $x_1x_2...x_{n^2+1}$. Let $t_i$ be the length of the longest increasing subsequence starting at $x_i$. If $t_i \geq n + 1$ for any $i$, then our claim is satisfied!

Otherwise, $t_i \leq n$ for all $i$, which allows us to map each of our $n^2 + 1$ $t_i$s to $n$ possible values. Then by the strong pigeonhole principle, we have that at least $n + 1$ of the $t_i$s have the same value.

Suppose $t_i = t_j, i < j$. If $x_i < x_j$, then $x_j$ is part of $x_i$’s increasing subsequence, so $t_i \neq t_j$ and we have a contradiction.

Thus, $x_i > x_j$ (they cannot be equal because all $x_i$s are distinct). This must be the case for all $i, j$ such that $t_i = t_j$; by the pigeonhole principle, there must exist a subsequence of length at least $n + 1$ which all start increasing subsequences of the same length. This subsequence must be decreasing because $x_i > x_j$ and $i < j$, so we have found a decreasing subsequence of length at least $n - 1$. 

$\square$