Overview

Sets Definitions (4.1–4.1.1)

Sets Operations (4.1.2–4.1.5)

Equality Proofs (4.1.6)

Venn Diagrams
Set Definition

Definition (informal): A set is a bunch/collection/group of objects.

Definition: The *elements* of the set are the objects contained in that set.

Sets can contain numbers, ordered sequences of numbers, strings, names, or other sets.

Objects are either *in* the set or *not in* the set. We don’t have a concept of an object being in a set multiple times. It’s a Boolean property.

We write curly braces around a comma-separated list to build a set.

Examples:

- \( H = \{ \text{Julie, Tyler, Julia} \} \)
- \( D = \{ \text{Boston Kreme, Glazed, Apple Crumb, Pumpkin} \} \)
- \( \mathbb{N} = \{ 0, 1, 2, 3, 4, \ldots \} \)
- \( S = \{ \text{Brown, Columbia, Cornell, \ldots, Yale} \} \)
Elements

- $H = \{ \text{Julie, Tyler, Julia} \}$
- $D = \{ \text{Boston Kreme, Glazed, Apple Crumb, Pumpkin} \}$
- $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots \}$
- $S = \{ \text{Brown, Columbia, Cornell, \ldots, Yale} \}$

Definition: We say say $x \in S$ if $x$ is an element of or in or a member of the set $S$.

- Julie $\in H$? Yes.
- Columbia $\notin H$.
- Columbia $\in S$? Yes.
- Dartmouth $\in S$? Yes.
Some Sets of Numbers

- $\emptyset = \{\} \text{ (empty set, null set)}$
- $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\} \text{ (non-negative integers)}$
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \text{ (integers)}$
- $\mathbb{Q} = \{1/2, -4/15, 21, \ldots\} \text{ (rationals)}$
- $\mathbb{R} = \{\sqrt{2}, -\pi, 21, \ldots\} \text{ (real numbers)}$
- $\mathbb{C} = \{i/2, 15 - i, \sqrt{7}, 21, \ldots\} \text{ (complex numbers)}$

Superscript plus limits to positive values: $\mathbb{Z}^+ = \mathbb{N}^+$.

Superscript minus limits to negative values: $21 \notin \mathbb{R}^-$.
Sets of sets

- \( A = \{1, 4, 9\} \)
- \( B = \{\{1, \{4\}\}, \{9\}\} \)

- \( 1 \in A? \) Yes.
- \( 1 \in B? \) No, but \( \{1, \{4\}\} \in B \).
- \( \exists x \in B, 1 \in x? \) Yes, \( x = \{1, \{4\}\} \in B \) and \( 1 \in x \).
Subsets

Definition: One set is a *subset* of another if every element of the first set is also an element of the second.

We write $S \subseteq T$ to say the set $S$ is a subset of set $T$. So, $S \subseteq T$ means $\forall x \in S, x \in T$.

Examples:

- $\mathbb{N} \subseteq \mathbb{Z}$? Yes, every positive integer is also a non-negative integer.
- $\mathbb{Z}^+ \subseteq \mathbb{N}$? Yes, every positive integer is also a non-negative integer.
- $\mathbb{C} \subseteq \mathbb{Z}$? No, $\mathbb{C} \not\subseteq \mathbb{Z}$. Some (many!) complex numbers are not integers. Although, $\mathbb{Z} \subseteq \mathbb{C}$.
- $\mathbb{N} \subseteq \mathbb{N}$. Yes, if sets are equal, all of the first must also be in the second!

Note: $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ looks a little bit like $3 \leq 4$.

We write $A \subset B$ to rule out equality (like $a < b$).
Non-Trichotomy

If $a$ and $b$ are integers, exactly one of these properties must hold:

- $a < b$
- $a = b$
- $a > b$

Not so for subsets. Example?

$A = \{0\}$
$B = \{1\}$

$A \subset B$? No, $0 \in A$, but $0 \notin B$.

$A = B$? No, they are different sets.

$A \supset B$? (superset!) No, $1 \in B$, but $1 \notin A$.  

Operations on sets: Union

- \( A = \{j, u, l, i, a\} \)
- \( B = \{j, u, l, i, e\} \)
- \( C = \{t, y, l, e, r\} \)

**Definition:** The *union* of sets \( X \) and \( Y \), \( X \cup Y \), consists of every element that is in either \( X \) or \( Y \). In other words, \( z \in X \cup Y \) means \( z \in X \) or \( z \in Y \).

**Example:** \( A \cup B = \{j, u, l, i, e, a\} \).
Operations on sets: Intersection

- $A = \{j, u, l, i, a\}$
- $B = \{j, u, l, i, e\}$
- $C = \{t, y, l, e, r\}$

Definition: The intersection of sets $X$ and $Y$, $X \cap Y$, consists of every element that is in both $X$ and $Y$. In other words, $z \in X \cap Y$ means $z \in X$ and $z \in Y$.

Example: $B \cap C = \{l, e\}$. 
Operations on sets: Set difference

- $A = \{j, u, l, i, a\}$
- $B = \{j, u, l, i, e\}$
- $C = \{t, y, l, e, r\}$

Definition: The set difference of sets $X$ and $Y$, $X - Y$, consists of every element that is in $X$ but not in $Y$. In other words, $z \in X - Y$ means $z \in X$ and $z \notin Y$.

Example: $C - A = \{t, y, e, r\}$.

Example: $A - B = \{a\}$. 
Operations on sets: Symmetric difference

- $A = \{j, u, l, i, a\}$
- $B = \{j, u, l, i, e\}$
- $C = \{t, y, l, e, r\}$

Definition: The symmetric difference of sets $X$ and $Y$, $X \triangle Y$, consists of every element that is in $X$ but not in $Y$ or in $Y$ but not $X$. In other words, $z \in X \triangle Y$ means $z \in X$ and $z \not\in Y$ or $z \in Y$ and $z \not\in X$.

Example: $C \triangle A = \{t, y, e, r, j, u, i, a\}$.

Example: $A \triangle B = \{a, e\}$. 
Operations on sets: Complement

- $A = \{j, u, l, i, a\}$
- $B = \{j, u, l, i, e\}$
- $C = \{t, y, l, e, r\}$

Definition: The complement of a set $X$, $\overline{X}$, is defined with respect to some universe of possible elements $U$. It consists of every possible element that is not $X$. In other words, $\overline{X} := U - X$.

Example: If $U$ is the universe of all letters in English, $\overline{A} = \{b, c, d, e, f, g, h, k, m, n, o, p, q, r, s, t, v, w, x, y, z\}$.

Example: If $U = \mathbb{Z}$, $\mathbb{Z}^- = \mathbb{Z}^+ - \{0\}$. 
Disjoint sets

Definition: Sets $A$ and $B$ are *disjoint* is they have no elements in common. In other words,

$$A \cap B = \emptyset \text{ or } A \subseteq \overline{B}.$$
Operations on sets: Power set

- $A = \{j, u, l, i, a\}$
- $B = \{j, u, l, i, e\}$
- $C = \{t, y, l, e, r\}$

Definition: The **power set** of a set $X$, $\mathcal{P}(X)$, is the set of all subsets of $X$. In other words, $\forall x \in \mathcal{P}(X), x \subseteq X$ and $\forall x \subseteq X, x \in \mathcal{P}(X)$.

Example: If $D = B \cap C = \{l, e\}$, $\mathcal{P}(D) = \{\emptyset, \{l\}, \{e\}, \{l, e\}\}$.

Example: If $E = A \cap B - C = \{j, u, i\}$, $\mathcal{P}(E) = \{\emptyset, \{j\}, \{u\}, \{i\}, \{j, u\}, \{j, i\}, \{u, i\}, \{j, u, i\}\}$.

Example: $\mathcal{P}(\emptyset) = \{\emptyset\}$. 
Operations on sets: Cardinality

- ▶ $A = \{j, u, l, i, a\}$
- ▶ $B = \{j, u, l, i, e\}$
- ▶ $C = \{t, y, l, e, r\}$

Definition: The *cardinality* of a set $X$, $|X|$, is the count of the number of unique elements in $X$.

Example: $|A| = |B| = |C| = 5$.

Example: $|\emptyset| = 0$.

Example: If $|A| = n$, $|\mathcal{P}(A)| = 2^n$. Each subset consists of a decision of whether to include or not include (2 possibilities) each of the $n$ elements of $A$. 
Building sets with predicates

General form: \( \{ \text{description of a set} \mid \text{filter on the set} \} \).

Examples:

- \( A ::= \{ n \in \mathbb{N} \mid n = 2k + 1 \text{ for some integer } k \} \)
- \( B ::= \{ x \in \mathbb{R} \mid x^2 > 1 \} \)

Note: Python has a notation for this idea.
Proving set equalities: Logic

$A = B$ means that, for all $x$, $x \in A$ if and only if $x \in B$. We can use this definition to prove various set equalities. Here’s a useful one.

Theorem: $A = B$ if and only if (iff) both $A \subseteq B$ and $B \subseteq A$.

$A = B$

iff for all $x$, $x \in A$ if and only if $x \in B$

iff for all $x$, if $x \in A$ then $x \in B$ and for all $x$, if $x \in B$ then $x \in A$

iff $\forall x \in A, x \in B$ and $\forall x \in B, x \in A$

iff $A \subseteq B$ and $B \subseteq A$. 
Proving set equalities: set-element method

We can use the previous result to prove other set equalities.

Theorem: For any sets \( A \) and \( B \) of elements in universe \( U \),
\[
A \cap B = \overline{A} \cup \overline{B}.
\]

“DeMorgan’s Law” relates intersection and union (“and” and “or”).

An object \( x \in A \cap B \) if it is \textit{not} in both \( A \) and \( B \). Such an element must either not be in \( A \) or not be in \( B \). It follows that such an element must be in \( \overline{A} \cup \overline{B} \). Thus, we have \( A \cap B \subseteq \overline{A} \cup \overline{B} \).

Note also that an object \( x \in \overline{A} \cup \overline{B} \) if it is either not in \( A \) or it is not in \( B \). Such an element can’t be in both \( A \) and \( B \), therefore. Said another way, it must be in \( \overline{A} \cap \overline{B} \). Thus, we have \( \overline{A} \cup \overline{B} \subseteq \overline{A} \cap \overline{B} \).

Since both \( \overline{A} \cap \overline{B} \subseteq \overline{A} \cup \overline{B} \) and \( \overline{A} \cup \overline{B} \subseteq \overline{A} \cap \overline{B} \) are true, we know \( A \cap B = \overline{A} \cup \overline{B} \).
Venn diagram translation

\[ (C - A) \cup B \]
\[ (C - A - B) \cup (C \cap B - A) \cup (A \cap B \cap C) \cup (A \cap B - C) \cup (B - A - C) \]